Completeness of $\mathbb{R}$

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Theorem. *Any Cauchy sequence in $\mathbb{R}$ converges.*

Proof. Let (x_n) be a Cauchy sequence in $\mathbb{R}$. Given $\varepsilon > 0$, there exists a positive integer N such that for all $m \geq N$ and $n \geq N$, we have $|x_n - x_m| < \varepsilon/2$. In particular we have $|x_n - x_N| < \varepsilon/2$ Or, equivalently,

$$x_n \in (x_N - \varepsilon/2, x_N + \varepsilon/2) \qquad \text{for all } n \geq N.$$

From this we make the following observations:
i) For all $n \geq N$, we have $x_n > x_N - \varepsilon/2$.
ii) If $x_n \geq x_N + \varepsilon/2$, then $n \in \{1, 2, \ldots, N-1\}$. Thus the set of n such that $x_n \geq x_N + \varepsilon/2$ is finite.

Let $S := \{x \in \mathbb{R} : \text{ there exists infinitely many } n \text{ such that } x_n \geq x\}$. We claim that S is nonempty, bounded above and that $\sup S$ is the limit of the given sequence.

From i) we see that $x_N - \varepsilon/2 \in S$. Hence S is nonempty.

From ii) it follows that $x_N + \varepsilon/2$ is an upper bound for S. That is, we claim that $y \leq x_N + \varepsilon/2$ for all $y \in S$. If this were not true, then there exists a $y \in S$ such that $x_n \geq y$ for infinitely many n. This implies that $x_n > x_N + \varepsilon/2$ for infinitely many n. This contradicts ii). Hence we conclude that $x_N + \varepsilon/2$ is an upper bound for S.

By the LUB axiom, there exists $\ell \in \mathbb{R}$ which is $\sup S$. As ℓ is an upper bound for S and $x_N - \varepsilon/2 \in S$ we infer that $x_N - \varepsilon/2 \leq \ell$. Since ℓ is the least upper bound for S and $x_N + \varepsilon/2$ is an upper bound for S we see that $\ell \leq x_N + \varepsilon/2$. Thus we have $x_N - \varepsilon/2 \leq \ell \leq x_N + \varepsilon/2$ or

$$|x_N - \ell| \leq \varepsilon/2.$$

For $n \geq N$ we have

$$\begin{aligned} |x_n - \ell| &\leq |x_n - x_N| + |x_N - \ell| \\ &< \varepsilon/2 + \varepsilon/2 = \varepsilon. \end{aligned}$$

We have thus shown that $\lim_{n\to\infty} x_n = \ell$.