Complex Analysis: Handout-2 (Results on Power Series)

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1 Results on Power Series

Theorem 1. *Let $\sum_{n=0}^{\infty} a_n(z-a)^n$ be a power series. There is a unique extended real number R, $0 \le R \le \infty$, such that the following hold:*
(i) *for all z with $|z-a| < R$, the series $\sum_{n=0}^{\infty} a_n(z-a)^n$ converges absolutely,*
(ii) *for all z with $|z-a| > R$, the series $\sum_{n=0}^{\infty} a_n(z-a)^n$ diverges.* □

Theorem 2. *Let $\sum_{n=0}^{\infty} a_n(z-a)^n$ be given and $0 < R \le \infty$ be its radius of convergence. Then for any r with $0 \le r < R$, the power series is uniformly convergent on the closed ball $B[a,r]$.* □

Corollary 3. *Any power series with positive radius of convergence is continuous in its disk of convergence.* □

Theorem 4 (Uniqueness Theorem for Power Series)**.** *Let $f(z) := \sum_n a_n(z-a)^n$ and $g(z) := \sum b_n(z-a)^n$ for all $z \in B(a,R)$ with $R > 0$. Assume that there exists a sequence (z_n) in $B(a,R)$ such that* (i) *$z_n \neq a$ for all n,* (ii) *$\lim z_n = a$ and* (iii) *$f(z_n) = g(z_n)$ for all n. Then $a_n = b_n$ for all n.*

Proof. For convenience, assume that $a = 0$. Since f and g are continuous at $a = 0$, $f(0) = \lim f(z_n) = \lim g(z_n) = g(0)$ and hence $a_0 = b_0$. Now proceed by induction. By induction hypothesis, we have $a_k = b_k$ for $0 \le k \le n$. We let $\varphi(z) := a_{n+1} + a_{n+2}z + \cdots$ and $\psi(z) := b_{n+1} + b_{n+2}z + \cdots$. Note that

$$\varphi(z) = \begin{cases} \frac{f(z) - \sum_{k=0}^{n} a_k z^k}{z^{n+1}} & \text{for } z \neq 0,\ |z| < R \\ a_{n+1} & \text{at } z = 0 \end{cases}$$

Similar remark applies to ψ. It follows that φ and ψ are continuous on $B(0,R)$. Consequently, $\varphi(z_r) = \psi(z_r)$ for all $r \in \mathbb{N}$. Hence we have

$$\varphi(0) = \lim \varphi(z_r) = \lim \psi(z_r) = \psi(0).$$

Since $\varphi(0) = a_{n+1}$ and $\psi(0) = b_{n+1}$, the result follows. □

Theorem 5. *Let $f(z) := \sum_{n=0}^{\infty} a_n z^n$ be a power series with radius of convergence $R > 0$. Then f is differentiable on $B(0, R)$. Furthermore, $f'(z) = \sum_{n=1}^{\infty} n a_n z^{n-1}$ for $z \in B(0, R)$. That is, a power series can be differentiated term by term in the disk of convergence.* □

Proposition 6. *Any power series function $f(z) := \sum a_n z^n$ is infinitely differentiable on its disk of convergence. In fact, we have*

$$f^{(k)}(z) = \sum_{n=k}^{\infty} n(n-1)\cdots(n-k+1) a_n z^{n-k}.$$

In particular, we have

$$a_n := \frac{f^{(n)}(0)}{n!}. \tag{1}$$

2 Analytic Functions

Definition 7. Let $U \subset \mathbb{C}$ be open. A function $f\colon U \to \mathbb{C}$ is said to be *analytic* in U if for any $a \in U$, there exists a sequence (a_n) of complex numbers and an $r > 0$ such that $B(a, r) \subset U$ and such that $f(z) = \sum_{n=0}^{\infty} a_n (z-a)^n$ for all $z \in B(a, r)$. The sequence (a_n) depends on a.

Example 8. Let $U = \mathbb{C}^*$ and $f(z) = 1/z$. Then for any $a \neq 0$, we have

$$f(z) = \frac{1}{a} \sum_{n=0}^{\infty} \left(-\frac{(z-a)}{a} \right)^n$$

for all $z \in B(a, |a|)$.

How did we get the power series? Look at the following:

$$\begin{aligned} \frac{1}{z} &= \frac{1}{(z-a)+a} \\ &= \frac{1}{a} \frac{1}{1 + \frac{z-a}{a}}. \end{aligned}$$

Thus, if we assume $|z-a| < |a|$, we can expand this as a geometric series.

Thus, f is analytic in U.

Ex. 9. $\exp\colon \mathbb{C} \to \mathbb{C}$ is analytic in $\mathbb{C}$.

Ex. 10. Analytic functions on U are holomorphic in U. In fact, analytic functions are infinitely differentiable.

Remark 11. It is a central result of Cauchy theory, the theme of Chapter 7, that the converse is also true. This is remarkable since this implies that any function which is once differentiable *on an open set* is infinitely differentiable. This should be contrasted with the existence of differentiable functions on intervals in $\mathbb{R}$ whose derivatives are not even continuous, leave alone being differentiable.

Our next theorem will show that any power series function on its disk of convergence is analytic on its domain. We need a preliminary result on iterated sum of a double series.

The next lemma gives us a sufficient condition under which the order of summation can be interchanged in a double series. First some notation. Let $a\colon \mathbb{N}\times\mathbb{N}\to\mathbb{C}$ be a function. We let $a_{ij} = a(i,j)$ for $(i,j)\in\mathbb{N}\times\mathbb{N}$. We want to investigate conditions under which

$$\sum_{i=1}^{\infty}\left(\sum_{j=1}^{\infty} a_{ij}\right) = \sum_{j=1}^{\infty}\left(\sum_{i=1}^{\infty} a_{ij}\right).$$

First of all notice that the displayed equation means a host of things: (1) $\sum_{j=1}^{\infty} a_{ij}$ is convergent, say, with sum α_i and that $\sum_{i=1}^{\infty}\alpha_i$ is convergent, (2) the series $\sum_{i=1}^{\infty} a_{ij}$ converges to, say, α'_j and $\sum_{j=1}^{\infty}\alpha'_j$ is convergent and (3) $\sum_i \alpha_i = \sum_j \alpha'_j$.

Note that we have neither defined the sum or convergence of a double series nor have shown that under the stated conditions the double series is convergent.

Lemma 12. *Let (a_{ij}) be a double sequence in $\mathbb{C}$, i.e. a function $a\colon \mathbb{N}\times\mathbb{N}\to\mathbb{C}$ so that $a(i,j) = a_{ij}$. Assume that $\sum_{j=1}^{\infty}|a_{ij}| = b_i$ for $i\in\mathbb{N}$ and that $\sum_i b_i$ is convergent. Then*

$$\sum_{i=1}^{\infty}\sum_{j=1}^{\infty} a_{ij} = \sum_{j=1}^{\infty}\sum_{i=1}^{\infty} a_{ij}.$$

Proof. This proof is due to Walter Rudin.

Note that the hypothesis implies that $\sum_i a_{ij}$ is absolutely convergent for every j fixed, since $|a_{ij}| \le \sum_j |a_{ij}| \le b_i$.

Consider the metric space $X := \{1/n : n\in\mathbb{N}\}\cup\{0\}$ with the usual metric. For each $i\in\mathbb{N}$, we define a function $f_i\colon X\to\mathbb{C}$ by setting

$$f_i(1/n) = \sum_{j\le n} a_{ij} \text{ and } f_i(0) = \sum_{j=1}^{\infty} a_{ij}.$$

It follows from hypothesis that f_i is continuous at 0 and that $\sum_i f_i$ converges uniformly on X to a function $g\colon X\to\mathbb{C}$ by the M-test. Hence g is continuous at 0. We have

$$\begin{aligned}
\sum_{i=1}^{\infty}\sum_{j=1}^{\infty} a_{ij} &= \sum_i f_i(0) = g(0) = \lim_{n\to\infty} g(1/n)\\
&= \lim_{n\to\infty}\sum_i f_i(1/n)\\
&= \lim_{n\to\infty}\sum_{i=1}^{\infty}\sum_{j=1}^{n} a_{ij}\\
&= \lim_{n\to\infty}\sum_{j=1}^{n}\sum_{i=1}^{\infty} a_{ij}\\
&= \sum_{j=1}^{\infty}\sum_{i=1}^{\infty} a_{ij}.
\end{aligned}$$

The last but one equality follows from the facts that $\sum_i a_{ij}$ is convergent and the algebra of convergent infinite series. $\square$

The next elementary exercise will be used in the proof of the next theorem.

Ex. 13. Show that if the order of a summation in a double series of the form $\sum_{j=0}^{\infty}\sum_{i=0}^{j} a_{ij}$ is reversed we get a double series of the form $\sum_{i=0}^{\infty}\sum_{j=i}^{\infty} a_{ij}$. *Hint:* Plot the terms a_{ij} at the points (i,j) of the lattice $\mathbb{Z}_+ \times Z_+$ in $\mathbb{R}^2$. Observe that the first iterated sum means to sum horizontally and then vertically. See Figure ???

Theorem 14. *Let $f(z) := \sum_{n=0}^{\infty} a_n(z-z_0)^n$ be a power series for $z \in B(z_0, R)$ with $R > 0$. Then f is analytic in $B(z_0, R)$.*

In fact, for any $a \in B(z_0, R)$ and $z \in B(a, R - |a|)$, we have

$$f(z) = \sum_{n=0}^{\infty} \frac{f^{(n)}(a)}{n!}(z-a)^n. \tag{2}$$

Proof. Assume $z_0 = 0$. Let $a \in B(0, R)$. We note that we need only show that f admits a power series expansion in powers of $(z-a)$. The explicit representation in (2) follows from the relation between the derivatives $f^{(n)}$ and the coefficients of the power series.

To write f in powers of $(z-a)$, we do the obvious thing. We have

$$\begin{aligned} f(z) &= \sum_{n=0}^{\infty} a_n((z-a)+a)^n \\ &= \sum_{n=0}^{\infty} a_n \sum_{k=0}^{n} \binom{n}{k}(z-a)^k a^{n-k} \\ &= \sum_{k=0}^{\infty}\left(\sum_{n=k}^{\infty}\binom{n}{k} a_n a^{n-k}\right)(z-a)^k. \end{aligned}$$

The only thing is to justify the last equality, i.e. to justify the interchange of the order of the sums.

By Lemma 12, this is valid, if $\sum_{n=0}^{\infty}\sum_{k=0}^{n} |a_n\binom{n}{k}a^{n-k}(z-a)^k|$ is convergent. But this series is dominated by $\sum_{n=0}^{\infty}|a_n|(|z-a|+|a|)^n$ which is convergent if $|z-a|+|a| < R$.

The displayed equation (2) now follows. $\square$