Arithmetic-Geometric Mean Inequality
Proof by Induction and Calculus

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Let $x_1, \dots, x_n$ be non-negative real numners. Their arithmetic mean and geometric mean are defined by

$$\mathrm{AM} := \frac{x_1 + \cdots + x_n}{n} \quad \text{and} \quad \mathrm{GM} := (x_1 \cdots x_n)^{1/n}.$$

The inequality of the title says that the arithmetic mean is greater than or equal to the geometric mean and equality holds iff all the x_i's are equal.

We prove this by mathematical induction and calculus. For $n = 1$, the statement holds true with equality.

Assume that the AM–GM statement is true for any set of n non-negative real numbers.

Let $n+1$ non-negative real numbers $x_1, \dots, x_{n+1}$ be given. We need to prove that

$$\frac{x_1 + \cdots + x_n + x_{n+1}}{n+1} - (x_1 \cdots x_n x_{n+1})^{\frac{1}{n+1}} \geq 0, \tag{1}$$

with equality only if all the n + 1 numbers are equal.

To avoid trivial cases, we may assume that all $n+1$ numbers are positive.

We consider the last number x_{n+1} as a variable and define the function

$$f(t) = \frac{x_1 + \cdots + x_n + t}{n+1} - (x_1 \cdots x_n t)^{\frac{1}{n+1}}, \qquad t > 0.$$

It suffices to show that $f(t) \geq 0$ for all $t > 0$, with $f(t) = 0$ only if $x_1, \dots, x_n$ and t are all equal. We employ the first and second derivative tests of calculus.

We have

$$f'(t) = \frac{1}{n+1} - \frac{1}{n+1}(x_1 \cdots x_n)^{\frac{1}{n+1}} t^{-\frac{n}{n+1}}, \qquad t > 0.$$

We are looking for points t_0 such that $f'(t_0) = 0$. Thus we obtain

$$(x_1 \cdots x_n)^{\frac{1}{n+1}} t_0^{-\frac{n}{n+1}} = 1.$$

That is, t_0 satisfies

$$t_0^{\frac{n}{n+1}} = (x_1 \cdots x_n)^{\frac{1}{n+1}}.$$

Or what is the same

$$t_0 = (x_1 \cdots x_n)^{\frac{1}{n}}.$$

That is, the only critical point t_0 of f is the geometric mean of $x_1, \ldots, x_n$. Note that if $t = R^n$ for very large $R \gg 1$, $f(t) \to \infty$ as $R \to \infty$. Hence it follows that f has a strict global minimum at t_0. Note that $f'' > 0$ and hence the function is convex. Hence t_0 must be a point of global minimum. We now compute $f(t_0)$.

$$\begin{aligned}
f(t_0) &= \frac{x_1 + \cdots + x_n + (x_1 \cdots x_n)^{1/n}}{n+1} - (x_1 \cdots x_n)^{\frac{1}{n+1}} (x_1 \cdots x_n)^{\frac{1}{n(n+1)}} \\
&= \frac{x_1 + \cdots + x_n}{n+1} + \frac{1}{n+1}(x_1 \cdots x_n)^{\frac{1}{n}} - (x_1 \cdots x_n)^{\frac{1}{n}} \\
&= \frac{x_1 + \cdots + x_n}{n+1} - \frac{n}{n+1}(x_1 \cdots x_n)^{\frac{1}{n}} \\
&= \frac{n}{n+1}\Big(\frac{x_1 + \cdots + x_n}{n} - (x_1 \cdots x_n)^{\frac{1}{n}}\Big).
\end{aligned}$$

The term withing brackets in the last step is non-negative in view of the induction hypothesis. The hypothesis also says that we can have equality only when $x_1, \ldots, x_n$ are all equal. In this case, their geometric mean t_0 has the same value, Hence, unless $x_1, \ldots, x_n, x_{n+1}$ are all equal, we have $f(x_{n+1}) > 0$. This completes the proof.