Exercises on Inverse images

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Ex. 1. Let $f\colon X \to Y$ be a constant map $f(x) = y_0$ for all $x \in X$. What is $f^{-1}(B)$ for $B \subset Y$?

Ex. 2. Let $f\colon \mathbb{R} \to \mathbb{R}$ be given by $f(x) = x^2$. What are $f^{-1}(1)$, $f^{-1}([0,1])$, $f^{-1}((0,1))$, $f^{-1}([-1,1])$, $f^{-1}([-4,4])$, $f^{-1}((-4,4))$, $f^{-1}([0,4])$ and $f^{-1}((0,4))$?

Ex. 3. Let $f\colon (0,\infty) \to (0,\infty)$ be given by $f(x) = 1/x$. What are $f^{-1}((0,1))$, and $f^{-1}((1,\infty))$?

Ex. 4. Let $f\colon \mathbb{R} \to \mathbb{R}$ be given by $f(x) := \sum_{k=0}^{n} a_k x^k$. Show that there exists a natural number N such that the number of elements in $f^{-1}(c)$ for any $c \in \mathbb{R}$ is at most N.

Ex. 5. Let $f\colon [-2\pi, 2\pi] \to \mathbb{R}$ be given by $f(x) = \sin x$. Find $f^{-1}([0,1])$.

Ex. 6. Let $f\colon \mathbb{R} \to \mathbb{R}$ be given by $f(x) = \cos x$. Find $f^{-1}(1)$.

Ex. 7. Let $f\colon \mathbb{R}^2 \to \mathbb{R}$ be given by $f(x,y) = x$. What are $f^{-1}(r)$ for $r \in \mathbb{R}$ and $f^{-1}([a,b])$? Draw pictures of these inverse images.

Ex. 8. Let $f\colon \mathbb{R}^2 \to \mathbb{R}$ be given by $f(x,y) = x^2 + y^2$. What are $f^{-1}(r)$ for $r \in \mathbb{R}$ and $f^{-1}([a,b])$? Draw pictures of these inverse images.

Ex. 9. Let $f\colon \mathbb{R}^2 \to \mathbb{R}$ be given by $f(x,y) = xy$. What are $f^{-1}(r)$ for $r \in \mathbb{R}$? Draw pictures of these inverse images.

Ex. 10. Let $f\colon M(n,\mathbb{R}) \to \mathbb{R}$ be given by $f(X) = \det(X)$. Identify the sets $f^{-1}(0)$ and $f^{-1}(\mathbb{R}^*)$, where $\mathbb{R}^*$ denotes the set of nonzero real numbers.

Ex. 11. Let $f\colon M(n,\mathbb{R}) \to M(n,\mathbb{R})$ be given by $f(X) = XX^T$. Identify the sets $f^{-1}(I)$.

Ex. 12. Let $f, g\colon M(n,\mathbb{R}) \to M(n,\mathbb{R})$ be given by $f(X) = X + X^T$ and $g(X) = X - X^T$. Identify the sets $f^{-1}(0)$ and $g^{-1}(0)$.

Ex. 13. Let $f\colon X \to Y$ and $g\colon Y \to Z$ be maps. Let $C \subset Z$. Show that $(g \circ f)^{-1}(C) = f^{-1}(g^{-1}(C))$.

Ex. 14. Let $f\colon X \to Y$ be a map. Let B_1, B_2, B be subsets of Y. Prove the following:

(a) $f^{-1}(B_1 \cup B_2) = f^{-1}(B_1) \cup f^{-1}(B_2)$.

(b) $f^{-1}(B_1 \cap B_2) = f^{-1}(B_1) \cap f^{-1}(B_2)$.

(c) Do (a) and (b) remain true if we deal with arbitrary unions and intersections?

(d) $f^{-1}(B^c) = (f^{-1}(B))^c$, where ^{c} denotes the complement in appropriate sets. In other words, $f^{-1}(Y \setminus B) = X \setminus f^{-1}(B)$.

(e) If $B_1 \subseteq B_2$, then $f^{-1}(B_1) \subseteq f^{-1}(B_2)$.

Ex. 15. Let $f\colon X \to Y$ be a map. Let $A \subset X$ and $B \subset Y$. Prove the following:
(a) $f(f^{-1}(B)) \subset B$.
(b) $A \subseteq f^{-1}(f(A))$.
(c) f is onto iff $f(f^{-1}(B)) = B$ for all $B \subset Y$.
(d) f is one-one iff $A = f^{-1}(f(A))$ for all $A \subset X$.

Ex. 16. Let $f\colon X \to Y$ be a bijection. Let $B \subseteq Y$. Show that $f^{-1}(B)$ is *the image* of B *under* the map f^{-1}.

Ex. 17. * Let $f\colon M(n,\mathbb{R}) \to M(n,\mathbb{R})$ be given by $f(X) = X^n$. Identify the sets $f^{-1}(0)$.

Ex. 18. * Let $f\colon \mathbb{R} \to \mathbb{R}$ be continuous and strictly increasing. Assume that $\alpha < \beta$ are in the image of f. What is $f^{-1}([\alpha, \beta])$?

Answer the same question if f is strictly decreasing.