Functional Analysis (2011) – Summary of Lectures

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Contents

I thank Ms. Jai Laxmi (a student of the course, Reg.No. 10MMMM34) for writing up a summary of my lectures. These notes are based on her contributions.

I rarely proof-read the notes. The best way you can thank me is to bring the mistakes in the Notes to my notice. (It is sad to record that I am rarely thanked!)

Does this mean: No mistakes? Far from truth!

We make references to the following books in the Course for specific results. The list may grow as we go along and hence they are not numbered.

Reference Books:

C.D. Aliprantis and O. Burkinshaw, *Principles of Real Analysis*, 3rd Edition, Harcourt Asia, (2000)
J. Bak and D.J. Newman, *Complex Analysis*, 2nd Edition, Springer Indian Reprint, (2009)
Bartle and Sherbert, *Introductory Real Analysis*, 3rd edition, Wiley International, (2001)
E. Kreyszig, *Introductory Functional Analysis with Applications*, Wiley Singapore Edition, (2001).
S. Kumaresan, *Topology of Metric Spaces*, Narosa, (2005).
S. Kumaresan, *Real Analysis – An Outline*, Unpublished Course Notes
(available at http://mtts.org.in/downloads)
B.V. Limaye, *Functional Analysis*, 2nd Edition, New Age International Ltd., (1996).
W. Rudin, *Real and Complex Analysis*, TMH Edition, 1973.

Throughout these notes, we let $\mathbb{K} = \mathbb{R}$ or $\mathbb{K} = \mathbb{C}$. We use the symbol $:=$, for example, $f(x) := x^2$ to say that the function f is defined by setting $f(x) = x^2$ for all x in the domain. This is same as writing $f(x) \stackrel{\text{def}}{=} x^2$. Can you guess what the symbol $x^2 =: f(x)$ means? $LHS =: RHS$ means that RHS is defined by LHS.

I started with the principle that a first course in functional analysis is meant first as a part of the general culture and second as an important tool for any future analyst. Hence the emphasis all through had been to look at concrete spaces of function and linear maps between them. This has two advantages: (1) the students get to see the typical applications of the results of functional analysis to other parts of analysis and (2) while dealing with such spaces and linear maps, more often than not they need to recall results concerning the kind of functions lying in the space and attend to the ε–δ–n_0 analysis. If we refuse to look at such examples, the only concrete examples students end up with are the sequence spaces, a situation not acceptable to people who look up to Functional Analysis as a bag of tools.

1. In Linear Algebra, you must have seen the concept of inner product spaces. If $(V, \langle\,,\,\rangle)$ is an IPS, then $\|x\| := (\langle x, x\rangle)^{1/2}$ is called the norm associated with the inner product. It satisfies the following properties.
   (a) $\|x\| \geq 0$ and $\|x\| = 0$ iff $x = 0$.
   (b) $\|\alpha x\| = |\alpha|\,\|x\|$ for all $\alpha \in \mathbb{K}$ and $x \in V$.
   (c) $\|x + y\| \leq \|x\| + \|y\|$ for all $x, y \in V$.

   The third property is proved using the Cauchy-Schwarz inequality.

2. Let X be a vector space over $\mathbb{K} = \mathbb{R}$ or $\mathbb{C}$. A *norm* $\|\ \|$ is a function $\|\ \| : X \to \mathbb{R}$ satisfying the following conditions:
   (a) $\|x\| \geq 0$ and $\|x\| = 0$ iff $x = 0$.
   (b) $\|\alpha x\| = |\alpha|\,\|x\|$ for all $\alpha \in \mathbb{K}$ and $x \in X$.
   (c) $\|x + y\| \leq \|x\| + \|y\|$ for all $x, y \in X$. This is known as the (*Triangle inequality*).

   The pair $(X, \|\ \|)$ is called a *normed linear space*

3. Finite dimensional examples.

   (a) We define $\|z\|_\infty := \max\{|z_j| : 1 \leq j \leq n\}$. We claim that $\|\ \|_\infty$ is a norm on X. We shall verify only the triangle inequality. Let $x, y \in \mathbb{K}^n$. Choose j, k such that $\|x\| = |x_j|$ and $\|y\| = |y_k|$. Then, for any $1 \leq r \leq n$,

   $$|x_r + y_r| \leq |x_r| + |y_r| \leq |x_j| + |y_k| = \|x\|_\infty + \|y\|_\infty .$$

   It follows that $\max\{|x_r + y_r| : 1 \leq r \leq n\} \leq \|x\|_\infty + \|y\|_\infty$. That is, $\|x + y\|_\infty \leq \|x\|_\infty + \|y\|_\infty$.

   (b) Let $X := \mathbb{K}^n$ and $z = (z_1, \ldots, z_n) \in X$. We then set $\|z\|_1 := \sum_{j=1}^n |z_j|$. It is an easy exercise to show that $(X, \|\ \|_1)$ is a normed linear space.

   (c) Let $X = \mathbb{K}^n$ and $\|x\| := (\sum_i |x_i|^2)^{1/2}$. Then $(X, \|\ \|)$ is a normed linear space. To prove triangle inequality, you need Cauchy-Schwarz. (See Item 82 for a proof.)

4. NLS of Functions. Compare each the three examples below with the earlier three.

   (a) This is the analogue of Item 3a. Let $S \neq \emptyset$. Let $B(S, \mathbb{K})$ denote the set of $\mathbb{K}$-valued bounded functions on S. Then $\|f\|_\infty := \sup_{x \in S}\{|f|(x)\}$. Then $(B(S, \mathbb{K}), \|\ \|_\infty)$ is an NLS. To derive the triangle inequality, mimic the argument in Item 3a:

   $$|f(x) + g(x)| \leq |f(x)| + |g(x)| \leq \|f\|_\infty + \|g\|_\infty .$$

   Hence $\sup\{|f(x) + g(x)| : x \in S\} \leq \|f\|_\infty + \|g\|_\infty$. Observe that this argument establishes two facts: (1) that $B(S, \mathbb{K})$ is closed under addition and (2) that $\|\ \|_\infty$ satisfies the triangle inequality.

(b) This is the analogue of Item 3b. Let $S = [a, b] \subset \mathbb{R}$. Let $X = C[a, b]$. Define $\| f \|_1 := \int_a^b |f(t)|\, dt$. Then $(C[a, b], \| \ \|_1)$ is an NLS.

The only thing that needs attention is the fact that if $f \in C[a, b]$ is such that $\int_a^b |f(t)|\, dt = 0$, then $f = 0$. We make use of the continuity of f to prove this. Assume $g \colon [a, b] \to \mathbb{R}$ is continuous and $g(t) \geq 0$ for $t \in [a, b]$. Assume further that $\int_a^b g(t)\, dt = 0$. We need to show that $g = 0$. Suppose not. Then there exists $s \in [a, b]$ such that $g(s) > 0$. For the sake of simplicity, assume that $s \in (a, b)$. For $\varepsilon = g(s)/2$, using the continuity of g, there exists $\delta > 0$ such that $(s - \delta, s + \delta) \subset (a, b)$ and for all $t \in (s - \delta, s + \delta)$, we have $g(t) \in (g(s) - \varepsilon, g(s) + \varepsilon)$, that is, $g(t) > g(s)/2$ for all such t. Hence

$$\int_a^b g(t)\, dt = \int_a^{s-\delta} g(t)\, dt + \int_{s-\delta}^{s+\delta} g(t)\, dt + \int_{s+\delta}^b f(t)\, dt \geq \int_{s-\delta}^{s+\delta} g(t)\, dt \geq g(s)\delta > 0,$$

a contradiction. A simple modification of argument takes care of the cases when $s = a$ or $s = b$.

(c) This is the analogue of Item 3c. Notation as in the last item. Define $\langle f, g \rangle := \int_a^b f(t)\overline{g(t)}\, dt$. This is an inner product. Let us define $\| f \|_2 := (\langle f, f \rangle)^{1/2}$. Then $(C[a, b], \| \ \|_2)$ is an NLS.

5. NLS of sequences. Note that each one below is an analogues of examples seen in Item 3.

(a) Look at $\ell^\infty := \{(z_n) : z_n \in \mathbb{K} \text{ and } (z_n) \text{ is bounded}\}$. Then $\ell^\infty \equiv B(\mathbb{N}, \mathbb{K})$. Define $\| (z_n) \|_\infty := \sup\{|z_n| : n \in \mathbb{N}\}$. Then $(\ell^\infty, \| \ \|_\infty)$ is an NLS.

(b) Let ℓ^1 be the set of all sequences (z_n) in $\mathbb{K}$ which are absolutely summable, that is, the associated series $\sum_k |z_k|$ is convergent. Then one easily shows that ℓ^1 is a vector space over $\mathbb{K}$. Define $\| (z_n) \| := \sum_n |z_n|$. Then $(\ell^1, \| \ \|_1)$ is an NLS.

(c) Let ℓ^2 be the set of all sequences (z_n) in $\mathbb{K}$ which are square summable, that is, the series $\sum_k |z_k|^2$ is convergent. It is not even clear why this is a vector space, especially why it is closed under addition. Define an "inner product" on this: for $z, w \in \ell^2$, let $\langle z, w \rangle := \sum_{k \in \mathbb{N}} z_k \overline{w_k}$. Why does $\langle z, w \rangle$ make sense? To show that the infinite series defining $\langle z, w \rangle$ converges, the first impulse is to show that it is absolutely convergent. That is, we wish to show that $\sum_k |z_k||\overline{w_k}|$ is convergent. Since this is a series of non-negative terms, we need to show that the sequence of partial sums is bounded above. Using the Cauchy-Schwarz inequality in $\mathbb{K}^n$, for each n, we arrive at the following:

$$\begin{aligned} \sum_{k=1}^n |z_k||\overline{w_k}| &= \sum_{k=1}^n |z_k||w_k| \\ &\leq \left(\sum_{k=1}^n |z_k|^2 \right)^{1/2} \left(\sum_{k=1}^n |w_k|^2 \right)^{1/2} \\ &\leq \left(\sum_{k=1}^\infty |z_k|^2 \right)^{1/2} \left(\sum_{k=1}^\infty |w_k|^2 \right)^{1/2}. \end{aligned} \tag{1}$$

Hence we conclude that the infinite series defining the inner product $\langle z, w \rangle$ is absolutely convergent and hence convergent. Note that (1) is the exact analogue of Cauchy-Schwarz inequality (on $\mathbb{K}^n$) for the inner product on ℓ^2!

Now we can show that ℓ^2 is closed under addition. To prove this we need to establish that $\sum_k |z_k + w_k|^2$ is convergent for $z, w \in \ell^2$. We have

$$\begin{aligned} \sum_k |z_k + w_k|^2 &\leq \sum_k |z_k|^2 + |w_k|^2 + 2\text{Re}\, z_k \overline{w_k} \\ &\leq \sum_k |z_k|^2 + \sum_k |w_k|^2 + 2\left(\sum_{k=1}^{\infty} |z_k|^2\right)\left(\sum_{k=1}^{\infty} |w_k|^2\right), \end{aligned} \tag{2}$$

where we have used the Cauchy-Schwarz inequality (1) to arrive at the last inequality. Hence $z + w \in \ell^2$. That it is closed under scalar multiplication is easy.

Thus we have shown that ℓ^2 is a vector space and $(z, w) \mapsto \langle z, w \rangle$ is indeed well-defined on ℓ^2. It is now a routine verification to show that $(z, w) \mapsto \langle z, w \rangle$ is an inner product on ℓ^2. If we now define $\| z \| := \langle z, z \rangle^{1/2}$, one easily shows $z \mapsto \| z \|$ is a norm on ℓ^2. Observe that (2) establishes the triangle inequality for the norm!

Note that we needed to establish the Cauchy-Schwarz inequality beforehand to show that ℓ^2 is a vector space and that $\langle , \rangle$ is an inner product on it!

6. Is there any containment relation between ℓ^1 and ℓ^2? We observed that ℓ^2 is not a subset of ℓ^1 but $\ell^1 \subset \ell^2$. For, $(1/k) \in \ell^2$ but $(1/k) \notin \ell^1$. If $z = (z_k) \in \ell^1$, choose $N \in \mathbb{N}$ such that $\sum_{k>N} |z_k| < 1$. The, for each $k > N$, we have $|z_k| < 1$ (Why?) so that $|z_k|^2 \leq |z_k|$. It follows that

$$\sum_{k=1}^{\infty} |z_k|^2 = \sum_{k=1}^{N} |z_k|^2 + \sum_{k>N} |z_k|^2 \leq \sum_{k=1}^{N} |z_k|^2 + \sum_{k>N} |z_k| < \infty.$$

We therefore conclude that $z \in \ell^2$.

7. Let X be a vector space over $\mathbb{K}$. Let $\{e_i : i \in I\}$ be a basis of X. Recall that this means that any $x \in X$ can be written *uniquely* as a finite linear combination of elements from the basis $\{e_i\}$:

$$x = \sum_i x_i e_i, \quad \text{where } \{i \in I : x_i \neq 0\} \text{ is finite.}$$

If we define $\| x \| := \sum_i |x_i|$ (a finite sum!), then $x \mapsto \| x \|$ is a norm on X. (Exercise)

Items 1–6 were done on 26-07-2011.

8. Let X be an NLS. Define $d(x, y) := \| x - y \|$. Then d is a metric on X, said to be induced by the norm. Note that $\| x \| = d(x, 0)$. All topological notions/concepts on a normed linear space are with respect to the metric d.

9. The metric induced by a norm is translation invariant: $d(x + a, y + a) = d(x, y)$, for $x, y, a \in X$.

10. Let X be an normed linear space. If A and $B$are subsets of X, we let

$$A + B := \{a + b : a \in A, b \in B\}.$$

If $A = \{a\}$, then we write $a + B$ in place of $A + B$ or $\{a\} + B$. Similarly, if $\lambda \in \mathbb{K}$, then $\lambda A := \{\lambda a : a \in A\}$.

We now claim that $B(x, r) = x + rB(0, 1)$. (Note that RHS makes sense!)

Let $y \in x + rB(0, 1)$. This means that y is of the form $y = x + ru$ where $u \in B(0, 1)$. Now,

$$f(y, x) = \|y - x\| = \|ru\| = r\,\|u\| < r, \text{ since } \|u\| < 1.$$

To see the other way inclusion, given $y \in B(x, r)$, we need to find $u \in B(0, 1)$ so that y can be written as $y = x + ru$. Work backwards. If such a u exists, then $ru = y - x$ so that $u = (y - x)/r$. We are through if we show that $u \in B(0, 1)$, that is, if $\|u\| < 1$. Since $y \in B(x, r)$, we do have $\|y - x\| < r$ and hence the claim.

11. A subset A of a metric space (X, d) is said to be *bounded* iff there exist $x \in X$ and $R > 0$ such that $A \subseteq B(x, R)$. Note that this is equivalent to the standard ubiquitous definition found in the the textbooks on metric spaces. For, if a nonempty A is bounded according to our definition, then for all $a, b \in A$, we have $d(a, b) \le d(a, x) + d(x, b) < 2R$ so that $\operatorname{diam}(A) := \sup\{d(a, b) : a, b \in A\} \le 2R$. Hence A is bounded according to the standard definition.

    In an NLS $(X, \|\ \|)$, we have a better criterion: A is bounded iff there exists $R > 0$ such that $\|a\| \le R$ for all $a \in A$. For, if A is bounded, then $A \subset B(x, R)$ for some $x \in X$ and $R > 0$. For any $a \in A$, we have $\|a\| \le \|a - x\| + \|x\|, < R + \|x\| =: R'$, say. Conversely, if $\|a\| \le R$ for all $a \in A$, then $A \subset B(0, R)$.

12. When is a vector subspace of a normed linear space bounded? Let V be a vector subspace of a normed linear space X. We claim that V is bounded iff $V = (0$. If $V = (0)$, then V is bounded. If V is nonzero and if $x \in V$ is nonzero, then $kx \in V$ for all $k \in \mathbb{N}$. But $\|kx\| = k\,\|x\|$. Therefore the set $\{k\,\|x\| : k \in \mathbb{N}\}$ is unbounded by the Archimedean principle. It follows that the subset $\{kx : k \in \mathbb{N}\} \subset V$ is unbounded and consequently V is unbounded.

13. What are the open subspaces of an normed linear space?

    If $Y \le X$ is a vector subspace of a normed linear space X, then Y is open iff $Y = X$. For, if Y is open, then there exists $r > 0$ such that $B(0, r) \subset Y$. Since Y is closed under scalar multiplication, $tB(0, r) \subset Y$ for any $t \in \mathbb{R}_+ := (0, \infty)$. That is, $\cup_{tk \in \mathbb{R}_+} tB(0, r) \subset Y$. We claim that the union $\cup_{t \in \mathbb{R}_+} tB(0, r) = X$. For if $x \in X$ is nonzero, let $u := x/\|x\|$ be the unit vector along the direction of x. Then $\frac{r}{2}u$ is of norm $r/2$ and hence it lies in $B(0, r)$. Hence $x = tu \in tB(0, r)$ where $t = \frac{2\|x\|}{r}$.

14. If we want to discuss continuity in the context of normed linear space, the obvious maps to consider are linear maps. Let $T\colon X \to Y$ be a linear map from a normed linear space X to another. Assume that T is continuous on X. This means that T is continuous at each $x \in X$. Since T is linear and maps $0 \in X$ to $0 \in Y$, it behooves us to consider the case when T is continuous at $0 \in X$.

    Assume that T is continuous at $0 \in X$. Given $\varepsilon > 0$ there exists $\delta > 0$ such that for all $x \in X$ with $\|x\| < \delta$, we have $\|Tx\| < \varepsilon$. Now given any nonzero $x \in X$, we consider the unit vector $u = x/\|x\|$ along the direction of x. Then $\left\|\frac{\delta}{2}u\right\| < \delta$ so that

$\left\| T(\frac{\delta}{2}u) \right\| < \varepsilon$. Using the linearity of T we obtain $\frac{\delta}{2\|x\|} \|Tx\| < \varepsilon$. Thus we have shown that $\|Tx\| \le C \|x\|$ for all $x \in X$ where $C = \frac{2\varepsilon}{\delta} \|x\|$.

We have struck a goldmine. Since T is additive, $T(x_1 - x_2) = T(x_1) - T(x_2)$ and hence

$$d(Tx_1, Tx_2) := \|Tx_1 - Tx_2\| = \|T(x_1 - x_2)\| \le C \|x_1 - x_2\| = Cd(x_1, x_2).$$

Thus we conclude that if the linear map T is continuous at 0, then T is Lipschitz, in particular, it is uniformly continuous on X!

15. The last item led us to the following theorem.

Theorem 1. *Let X and Y be normed linear space's. Let $T\colon X \to Y$ be a linear map. Then the following are equivalent.*
(i) *T is continuous on X.*
(ii) *T is continuous at 0.*
(iii) *There exists $C > 0$ such that for all $x \in X$, we have $\|Tx\| \le C \|x\|$.*
(iv) *T is Lipschitz continuous.*
(v) *T is uniformly continuous.* □

16. Exercise. Show that a linear map is continuous iff there exist $a \in X$, $r > 0$ and $M > 0$ such that $\|Tx\| \le M$ for all $x \in B(a, r)$.

17. Any linear map from $(\mathbb{K}^n, \| \ \|_p)$, $p = \infty, 1, 2$, to **any** normed linear space Y is continuous. For, if $\{e_k\}_1^n$ is the standard basis of $\mathbb{K}^n$, then we write $x = \sum_k x_k e_k$ so that $\|Tx\| \le \sum_k |x_k| \|Te_k\|$. Let $C := \max\{\|Te_k\| : 1 \le k \le n\}$. We have

$$\|Tx\| \le C \sum_k |x_k| \le \begin{cases} C \|x\|_1 & \text{if } p = 1 \\ C\sqrt{n} \|x\|_2 & \text{if } p = 2 \\ Cn \|x\|_\infty & \text{if } p = \infty. \end{cases}$$

We have applied Cauchy-Schwarz inequality to the sum $\sum_k |x_k| = \sum_k (|x_k| \cdot 1)$ to arrive at the second inequality above,

18. There exist linear maps between normed linear spaces which are not continuous.

(a) Consider $\mathbf{c}_{00} \subset \ell^\infty$ be the set of all sequences whose terms are zero all but finitely many. Thus $(z_n) \in \mathbf{c}_{00}$ iff there exists $N \in \mathbb{N}$ such that $z_n = 0$ for $n \ge N$. Define $\|z\| = \|z\|_\infty$. Define a linear map T by setting $Tz = (1z_1, 2z_2, 3z_3, \ldots, nz_n, \ldots)$. Note that $\|Te_n\| = n$. Hence there exists no C such that $\|Tz\| \le C \|z\|$. Thus T is linear but not continuous.

(b) Consider the vector subspace $C^1[0,1]$ of continuously differentiable functions on $[0,1]$ of $C[0,1]$ under the sup norm $\| \ \|_\infty$. Let $T = D$ be the linear map $T\colon (C^1[0,1], \| \ \|_\infty) \to (C[0,1], \| \ \|_\infty)$ defined by $Tf = f'$, the derivative of f. Then T is linear. If we consider the polynomial functions $f_n(t) = t^n$, then $\|f_n\|_\infty = 1$ and $\|Tf\|_\infty := \sup\{nt^{n-1} : t \in [0,1]\} = n$ for $n \in \mathbb{N}$. Hence there exists no C such that $\|Tf\|_\infty \le C \|f\|_\infty$ for $f \in C^1[0,1]$. Thus, we conclude that T is not continuous.

Note that if we restrict T to the space of polynomials, this example is "essentially the same" as the last one!

19. Let $a \in \ell^\infty$ and $z \in \ell^1$. Then $Tz := \sum_k a_k z_k$ is linear and continuous from ℓ^1 to $\mathbb{K}$. For, $\|Tz\| \leq \|a\|_\infty \cdot \|z\|_1$.

20. Consider $X := C[a, b]$ but with two different norms $\|f\|_\infty := \sup\{|f(t)| : t \in [a, b]\}$ and $\| \ \|_1$ as in Item 4.c. We have an obvious linear mapping from the normed linear space $(X, \| \ \|_\infty)$ to $(X, \| \ \|_1)$, namely, the identity map $T : f \mapsto f$. Is it continuous? In view of the theorem in Item 15, it suffices to show that $\|Tf\|_1 \equiv \|f\|_1 \leq C \|f\|_\infty$ for some $C > 0$. This is easy:

$$\|f\|_1 = \int_a^b |f(t)| \, dt \leq \int_a^b \|f\|_\infty \, dt = (b - a) \|f\|_\infty .$$

It follows that the identity map from $(X, \| \ \|_\infty)$ to $(X, \| \ \|_1)$ is continuous.

Question: Is the identity map from $(X, \| \ \|_1)$ to $(X, \| \ \|_\infty)$ continuous? Think geometrically. What is the geometric meaning of the integral of a non-negative function?

21. It must be now clear to the readers that to check whether or not a linear map $T : X \to Y$ between two normed linear spaces is continuous, we need to estimate $\|Tx\|$ in terms of $\|x\|$.

Items 8–19 were done on 27-07-2011.

22. **Young's Inequality.** Let $p > 0$, $q > 0$ be such that $(1/p) + (1/q) = 1$. If $p = 1$, we let $q = \infty$. Young's inequality holds:

$$(x^p/p) + (y^q/q) \geq xy \text{ for all } x > 0 \text{ and } y > 0. \tag{3}$$

Assume $p > 1$. The equality holds iff $x^{p-1} = y$ iff $x^{1/q} = y^{1/p}$.

p and q, related as above, are called conjugate indices.

Fix y. Consider $f(x) = (x^p/p) + (y^q/q) - xy$ and apply the one variable derivative tests for extrema.

Details: $f'(x) = x^{p-1} - y = 0$ iff $x^{p-1} = y$. Let $a := y^{1/(p-1)}$. Then $f''(x) > 0$ so that $x = a$ is a point of minimum. Since

$$f(a) = \frac{y^{\frac{p}{p-1}}}{p} + \frac{y^q}{q} - y^{1+\frac{p}{p-1}} = y^q \left(\frac{1}{p} + \frac{1}{q} \right) - y^q = 0,$$

it follows that $f(x) \geq f(a) = 0$, that is, the said inequality is established.

For $p = 1$, the result is obvious. Note also, for $p > 1$, the equality occurs iff $x^{p-1} = y$.

23. **Hölder's Inequality.** Let X be $\mathbb{K}^n$ and, for $1 \leq p < \infty$, let $\|x\|_p := \left(\sum_i |x_i|^p\right)^{1/p}$ and for $p = \infty$, let $\|x\|_\infty := \max\{|x_i| : 1 \leq i \leq n\}$. For $p > 1$, let q be such that $(1/p) + (1/q) = 1$. For $p = 1$ take $q = \infty$. We have **Hölder's inequality:**

$$\sum_i |a_k||b_k| \leq \|a\|_p \, \|b\|_q \,, \text{ for all } a, b \in \mathbb{K}^n.$$

Hint: Take $x = \frac{|a_k|}{\|a\|_p}$ and $y = \frac{|b_k|}{\|b\|_q}$ in Young's inequality (3) and sum over k.

When does equality occur?

24. **Minkowski inequality.** $(\mathbb{K}^n, \| \ \|_p)$ is a normed linear space for $1 \leq p \leq \infty$. $\| \ \|_p$ is called the L^p-norm on $\mathbb{K}^n$.

For $1 < p < \infty$, observe

$$\begin{aligned} \sum_i |a_i + b_i|^p &= \sum_i |a_i + b_i||a_i + b_i|^{p-1} \\ &\leq \sum_i |a_i||a_i + b_i|^{p-1} + \sum_i |b_i||a_i + b_i|^{p-1}. \end{aligned}$$

Apply Holder's inequality to each of the summands. The triangle inequality for $\| \ \|_p$ is called **Minkowski inequality**.

25. ℓ^p spaces. The ℓ^1, ℓ^2 and ℓ^∞ spaces seen earlier in Item 5 are three of the family of normed linear spaces defined as follows. Let $1 \leq p < \infty$. We define

$$\ell^p \ := \ \{(z_n) : \sum_n |z_n|^p < \infty\} \ \ (1 \leq p < \infty).$$

For $p = \infty$, ℓ^∞ and the norm $\| \ \|_\infty$ are as defined earlier. By now the reader must be able to show that ℓ^p is a normed linear space withe the norm $\|(z_n)\| := (\sum_n |z_n|^p)^{1/p}$, for $1 \leq p \leq \infty$.

26. Let $\| \ \|$ and $| \ |$ be two norms on an normed linear space X. Let $\mathcal{T}_1$ and $\mathcal{T}_2$ be the topologies induced by these norms. We say that $\| \ \|$ and $| \ |$ are *equivalent* on X if these two topologies are the same. Note that this means that the identity map $\mathrm{Id}\colon (X, \| \ \|) \to (X, | \ |)$ is a homeomorphism.

27. Last item and the theorem in Item 15 leads us to the following important and useful characterization of equivalent norms.

Two norms $\| \ \|_1$ and $\| \ \|_2$ on a vector space X are equivalent iff there exist (positive) constants C_1 and C_2 such that

$$C_1 \|x\|_1 \leq \|x\|_2 \leq C_2 \|x\|_1 \,, \text{ for all } x \in X.$$

In fact, almost all text books define equivalent norms using this characterization.

28. Extremely useful observations: Let $\| \ \|_1$ and $\| \ \|_2$ be two equivalent norms on a vector space X.

    (a) A sequence (x_n) in X converges to an x w.r.t. $\| \ \|_1$ iff it does so with respect to $\| \ \|_2$.
    (b) A sequence (x_n) in X is Cauchy in $(X, \| \ \|_1)$ iff it is Cauchy in $(X, \| \ \|_2)$.
    (c) $(X, \| \ \|_1)$ is complete iff $(X, \| \ \|_2)$ is complete.
    (d) If we want to estimate some object in X, we can use whichever of the two norms is convenient to us. In particular, a subset of X is bounded in $\| \ \|_1$ iff it is bounded in $\| \ \|_2$.

29. **An aside:** If we contemplate an analogous definition for the equivalence of metrics, we need to be more careful. For, the metric space $X = [1, \infty)$ with the standard metric $d(x, y) := |x - y|$ is complete. But if we use the homeomorphism $x \mapsto 1/x$ of $(0, 1]$ onto $[1, \infty)$ to define a new metric d_1 on X by setting $d_1(x, y) := |\frac{1}{x} - \frac{1}{y}|$, then d and d_1 induce the same topology on X. But (X, d) is complete while (X, d_1) is not!

    So, a more meaningful definition of equivalence of metrics would be as follows. Two metrics d_1 and d_2 are *equivalent* if there exist constant C_1 and C_2 such that

$$C_1 d_1(x, y) \leq d_2(x, y) \leq C_2 d_1(x, y) \text{ for } x, y \in X.$$

30. The following result (or more precisely, the result of Item 31) will be used many times in the course.

    Theorem 2. *Any two norms on $\mathbb{K}^n$ are equivalent.*

    Proof. It suffices to show that a given norm on $\mathbb{K}^n$ is equivalent to the Euclidean norm. (Why? The notion that two norms being equivalent is an equivalence relation among the norms on the same vector space.)

    Let η denote a norm on $\mathbb{K}^n$. We continue to denote the Euclidean norm by $\| \ \|$. The norms are equivalent iff there exist positive constants C_1, C_2 such that

$$C_1 \|x\| \leq \eta(x) \leq C_2 \|x\| \text{ for all } x \in \mathbb{R}^n. \tag{4}$$

    Let $\{e_k : 1 \leq k \leq n\}$ be the standard basis of $\mathbb{K}^n$. Then we have for any $x = (x_1, \dots, x_n) = \sum_{k=1}^n x_k e_k$,

$$\begin{aligned} \eta(x) \equiv \eta\left(\sum_{k=1}^n x_k e_k\right) &\leq \sum_{k=1}^n |x_k| \, \eta(e_k) \\ &\leq M \sum_{k=1}^n |x_k| \\ &\leq M \sum_{k=1}^n \|x\| \\ &= Mn \|x\|, \end{aligned} \tag{5}$$

    where $M := \max\{\eta(e_k) : 1 \leq k \leq n\}$. Thus the right most inequality in (4) is obtained with $C_2 := Mn$. (An aside: One can, in fact, improve this constant to $M\sqrt{n}$, by applying the Cauchy-Schwarz inequality to the sum $\sum_{j=1}^n |x_j| \, y_j$ where $y_j = 1$ for all $1 \leq j \leq n$. See Item 17.)

    To get the left most inequality of (4), we make some preliminary observation. If $\|x\| = 1$, what the left most inequality means is that

$$\eta(x) \geq C_1 \text{ for all } x \in S := \{x \in \mathbb{R}^n : \|x\| = 1\}.$$

    If $x \in S$, then $\|x\| = 1$ and hence $x \neq 0$. It follows that the map satisfies: $x \mapsto \eta(x) > 0$ for $x \in S$. What we want to claim is that it is bounded below by a positive constant

C_1. This triggers our memory. We have seen something similar in Real Analysis. So what we need to show is that $\eta : S \to \mathbb{R}$ is a continuous function on the compact subset $S \subset (\mathbb{K}^n, \| \ \|)$. But we have done it already. The inequality (5) shows that η is (Lipschitz) continuous from $(S, \| \ \|)$ to $\mathbb{R}$:

$$\left|\eta(x) - \eta(y)\right| \le \eta(x-y) \le Mn \|x-y\|.$$

Hence the function η restricted to S attains the minimum value, say, C_1 at some point $x_0 \in S$. Hence $\eta(x) \ge C_1$ for all $x \in S$. Consequently, $\eta\left(\frac{x}{\|x\|}\right) \ge C_1$ for all nonzero $x \in \mathbb{K}^n$. It follows that $\eta(x) \ge C_1 \|x\|$ for all $x \in \mathbb{K}^n$. □

Note the crucial role played by the local compactness of the spaces $\mathbb{K}^n$ when we made use of the fact that the unit sphere in $\mathbb{K}^n$ is compact.

31. Extend the last result to any finite dimensional vector space X: Any two norms on a finite dimensional vector space are equivalent.

    Many students had problem in writing a careful proof of this. So, here we go. Let $(X, \| \ \|)$ be an n dimensional normed linear space over $\mathbb{K}$. Fix a basis $\{v_k : 1 \le k \le n\}$ of X. Let $T\colon \mathbb{K}^n \to X$ be the unique linear map $Te_k = v_k$, where $\{e_k\}$ is the standard basis of $\mathbb{K}^n$. By Item 17, T is continuous and a linear isomorphism. Hence if we set $|z| := \|Tz\|$ for $z \in \mathbb{K}^n$, then $| \ |$ is a norm on $\mathbb{K}^n$ and hence equivalent to the Euclidean norm on $\mathbb{K}^n$ by the last item. We can therefore find C_1 and C_2 such that

    $$C_1 \|z\| \le |z| \le C_2 \|z\|, \quad \text{for } z \in \mathbb{K}^n.$$

    Using T^{-1}, we define a norm $p\colon x \mapsto \left\|T^{-1}x\right\|$ on X. The inequalities above say that the original norm $\| \ \|$ on X and p are equivalent:

    $$C_1 \left\|T^{-1}(x)\right\| \le |T^{-1}(x)| \le C_2 \left\|T^{-1}(x)\right\| \quad \text{for } Tz = x \in X.$$

    Or, we have:

    $$C_1 p(x) \le \|x\| \le C_2 \|x\|, \text{ since } |T^{-1}(x)| = \left\|T(T^{-1}(x)\right\|.$$

    That is, any norm on X is equivalent to the norm p.

Items 22–31 were done on 28-07-2011.

32. Recall the following results from the topology of metric spaces. Let (X, d) be a metric space.

    (a) Let $A \subset X$. If the metric space $(A, d|_A)$ is complete, then A is closed in X.

    (b) Assume that (X, d) is complete. Then A is closed in X iff $(A, d|_A)$ is complete.

33. Two most important consequences/corollaries of Item 30 (Item 31) are formulated as the next theorem.

    Theorem 3. 1) *Any finite dimensional normed linear space is complete.*
    2) *If Y is a finite dimensional vector subspace of a normed linear space X, then Y is closed in X.*

Proof. We use the notation established in Item 31. Then $T^{-1}\colon (X,p) \to (\mathbb{K}^n, \| \ \|)$ is a linear isomorphism which is also an isometry: $\left\| T^{-1}(x) \right\| = p(x)$. If (x_n) is Cauchy in $(X, \| \ \|)$, then it is Cauchy in p (since the norms $\| \ \|$ and p on X are equivalent) and hence the sequence $(T^{-1}(x_n))$ is Cauchy in $(\mathbb{K}^n, \| \ \|)$. $(\mathbb{K}^n, \| \ \|)$ is complete and hence the sequence $(T^{-1}(x_n))$ is convergent. Let $z = \lim T^{-1}(x_n)$ and $x := Tz$. Then $p(x - x_n) = \left\| T^{-1}(x_n) - z \right\| \to 0$. Since the norms $\| \ \|$ and p on X are equivalent, it follows that $\| x_n - x \| \to 0$. Hence (1) is proved.

(2) follows from (1) and the result quoted in Item 32b. □

34. The proof also leads us to an interesting proof of the following result of Item 17.

    Theorem 4. *Any linear map T from a finite dimensional normed linear space X to any normed linear space Y is continuous.*

    Consider a new norm $|x| := \| x \| + \| Tx \|$ on X. Then $\| \ \|$ and $| \ |$ are equivalent. Hence $\| TX \| \leq |x|$ establishes the continuity of T!

35. A corollary of the last item 34 is: If two normed linear space's X and Y have the same dimension, then X and Y are homeomorphic.

    Let X and Y be normed linear spaces of dimension n over $\mathbb{K}$. Let $T\colon X \to Y$ be a linear isomorphism. By the last item both T and T^{-1} are continuous.

36. A complete normed linear space is called *Banach* space.

    We shall look at various normed linear spaces and check whether they are complete. In each the case, it will turn out to be crucial to understand when a sequence is convergent as concretely as possible. For example, in the case of $\mathbb{K}^n$, under any norm a sequence (v_n) converges to v iff the sequence $(z_{nk})_n$ of k-th coordinates converges to z_k, the k-th coordinate of v. Hence we easily conclude that $\mathbb{K}^n$ is complete.

    In the case of the normed linear space $(B(S,\mathbb{K}), \| \ \|_\infty)$, the convergence of (f_n) to f is the same as the uniform convergence of (f_n) to f on S.

37. The normed linear space $(B(S,\mathbb{K}), \| \ \|_\infty)$ is complete.

    Let a Cauchy sequence (f_n) be given. We claim that the sequence of functions (f_n) is uniformly Cauchy on S. Given $\varepsilon > 0$, let $N \in \mathbb{N}$ be such that for $m, n \geq N$ we have $\| f_n - f_m \|_\infty < \varepsilon$. We observe, for $n, m \geq N$,

    $$\begin{aligned} \| f_n - f_m \|_\infty < \varepsilon &\implies \sup\{|f_n(x) - f_m(x)| : x \in S\} < \varepsilon \\ &\implies |f_n(x) - f_m(x)| < \varepsilon \text{ for } x \in S. \end{aligned}$$

    Fix $x \in S$. The sequence of scalars $(f_n(x))$ is a Cauchy sequence in $\mathbb{R}$ (or in $\mathbb{C}$, if we are dealing with complex valued functions). For definiteness sake, we shall assume that we are working with real valued functions. (The proof for complex valued functions is exactly the same.)

    Since $\mathbb{R}$ is complete, the sequence converges to a real number $r \in \mathbb{R}$, which we denote by $f(x)$, to show the dependence of $\alpha = \lim_{n\to\infty} f_n(x)$ on x. Thus, for each $x \in S$, we get a real number $f(x)$ such that $\lim_{n\to\infty} f_n(x) = f(x)$. (What we have shown so far is that the sequence f_n converges to some function $f\colon X \to \mathbb{R}$ pointwise.) To show that $f_n \to f$ in the metric space, we first of all need to show that the function $x \mapsto f(x)$ lies in $B(X)$, that is, f is bounded on X. Next we need to show that $f_n \to f$ in the metric space, that is, we need to prove that $f_n \to f$ uniformly on X. If we accomplish these two tasks, the proof is complete.

We first show that f is bounded. Since (f_n) is Cauchy in the metric space, it is bounded. Therefore, there exists $M > 0$ such that $\| f_n \| \leq M$ for $n \in \mathbb{N}$. In particular, $|f_n(x)| \leq M$ for all $x \in S$ and $n \in \mathbb{N}$. Fix $x \in S$. Since $f_n(x) \to f(x)$, given $\varepsilon = 1$, there exists N (which may depend on x and so we may denote it by $N(x)$) such that $|f_n(x) - f(x)| < 1$ for $n \geq N(x)$. It follows that

$$\begin{aligned}
|f(x)| &= \left|[f(x) - f_{N(x)}(x)] + f_{N(x)}(x)\right| \\
&\leq \left|f(x) - f_{N(x)}(x)\right| + \left|f_{N(x)}(x)\right| \\
&< 1 + M.
\end{aligned}$$

We have therefore shown that $|f(x)| \leq M + 1$ for all $x \in S$ and hence $\| f \| \leq M + 1$. We conclude that f is bounded and hence is an element of $B(X)$. (Go through the argument carefully, as we shall again use it.)

We now show that $f_n \to f$ uniformly on S. (This proof is delicate and see the remark after the proof.) Let $\varepsilon > 0$ be given. Since (f_n) is Cauchy in the metric, there exists $n_0 \in \mathbb{N}$ such that $\| f_n - f_m \|_\infty < \varepsilon/2$ for $m, n \geq n_0$. We claim that $|f(x) - f_n(x)| < \varepsilon$ for $n \geq n_0$.

Fix $x \in X$. Since $f_n(x) \to f(x)$, for the given $\varepsilon > 0$, there exists $N(x) \in \mathbb{N}$ such that $|f_m(x) - f(x)| < \varepsilon/2$ for $m \geq N(x)$. We have, for all $n \geq n_0$,

$$\begin{aligned}
|f(x) - f_n(x)| &= |f(x) - f_m(x) + f_m(x) - f_n(x)| \\
&\qquad \text{(for any } m \in \mathbb{N}) \\
&= |f(x) - f_m(x) + f_m(x) - f_n(x)| \\
&\qquad \text{(in particular for any } m \in \mathbb{N} \text{ with } m \geq N(x)) \\
&= |f(x) - f_m(x) + f_m(x) - f_n(x)| \\
&\qquad \text{(for any } m \in \mathbb{N} \text{ with } m \geq N(x) \text{ and } m \geq n_0) \\
&< \varepsilon/2 + \varepsilon/2 = \varepsilon.
\end{aligned}$$

That is, $f_n \to f$ uniformly on x. □

Remark: Please go through the proof once more, which will be referred to as curry-leaves trick. The integer m in the above is the curry leaf of the trick.

38. Let S be a compact space. Then $(C(S), \| \ \|_\infty)$ is complete. Use the last item and Item 32b. Recall the result from real analysis which says that the uniform limit of continuous functions is continuous.

39. Let $C[0,1]$ be endowed the the L^1 norm $\| \ \|_1$, as in Item 4b. Then the normed linear space $(C[0,1], \| \ \|_1)$ is not complete.

    We show that $(C[0,1], \| \ \|_1)$ is not complete. Most often a naive guess is to consider the sequence (x^n) in $(C[0,1], \| \ \|_1)$. It is easy to see that this sequence does converge to the zero function in $\| \ \|_1$. So what we need is to start with a function which is discontinuous, say, at an interior point and which can be the 'limit' of a sequence of continuous functions. To make the notation simpler, we shall show that $(C[-1,1], \| \ \|_1)$ is not complete.

The strategy is as follows. Construct a continuous function f_n which is 0 on $[-1,0]$, 1 on $[1/n,1]$ and linear on $[0,1/n]$. This is Cauchy and it does not converge to a continuous function in $\| \ \|_1$. This is a subtle step in the proof.

We now give the details. First we write down the function explicitly:

$$f_n(x) := \begin{cases} 0 & \text{for } x \in [-1,0] \\ nx & \text{for } x \in (0,1/n] \\ 1 & \text{for } x \in [1/n,1] \end{cases}$$

Draw pictures of f_n's. If you look at the picture and recall the geometric meaning of $\| \ \|_1$, then it is geometrically clear that $\| f_n - f_m \|_1 \to 0$ as $m,n \to \infty$. We shall give explicit estimate for $\| f_n - f_m \|_1$ to quell your doubts, if any. For $n > m$, we have

$$\begin{aligned} \| f_n - f_m \|_1 &= \int_{-1}^{0} |f_n - f_m| + \int_0^{1/n} |f_n - f_m| \\ &\quad + \int_{1/n}^{1/m} |f_n - f_m| + \int_{1/m}^{1} |f_n - f_m| \\ &= I_1 + I_2 + I_3 + I_4. \end{aligned}$$

Clearly, $I_1 = 0$ and so is I_4. Let us look at I_2:

$$I_2 := \int_0^{1/n} (n-m)x\,dx = \frac{(n-m)}{2n^2} \le \frac{n}{2n^2} = \frac{1}{2n}.$$

Now let us estimate I_3. First we observe that $f_n(x) = 1$ for $1/n \le x \le 1$. Hence,

$$\begin{aligned} I_3 := \int_{1/n}^{1/m} (1 - mx)\,dx &\le \int_{1/n}^{1/m} 1\,dx \qquad (6) \\ &\le \frac{1}{m} - \frac{1}{n} \\ &= \frac{n-m}{mn} \\ &\le \frac{n}{mn} = \frac{1}{m}. \qquad (7) \end{aligned}$$

So, given $\varepsilon > 0$, if we choose $N > 1/\varepsilon$ and assume that $n > m \ge N$, the inequality (7) shows that $\| f_n - f_m \|_1 < \varepsilon$ for $m,n \ge N$. Hence the sequence (f_n) is Cauchy. (We wanted to give precise ε-N argument. In fact, we could have stopped at (6), since the sequence $(1/n)$ is convergent and hence is Cauchy!)

Now we show that the sequence (f_n) is not convergent in the space $(C[-1,1], \| \ \|_1)$. Let f_n converge to f in the space. We then have

$$\begin{aligned} \| f - f_n \|_1 &= \int_{-1}^{0} |f - f_n| + \int_0^{1/n} |f - f_n| + \int_{1/n}^{1} |f - f_n| \\ &= J_1 + J_2 + J_3 \to 0, \end{aligned}$$

as $n \to \infty$. Hence each of the terms (being a sum of nonnegative terms) goes to zero. Let us look at J_1. Since $f_n = 0$ on $[-1, 0]$, we see that $J_1 = \int_{-1}^{0} |f|$. This is independent of n and saying that this goes to zero as $n \to \infty$ is same as saying that this constant is zero. Hence (by an argument we have seen in Item 4b, $|f|$ and hence f is zero on $[-1, 0]$. Next consider J_3. We have $J_3 \to 0$ as $n \to \infty$. We fix N. Then for any $n \geq N$, we have $f_n(x) = 1$ on $[1/N, 1]$. Hence

$$J_3 \geq \int_{1/N}^{1} |f - f_n| = \int_{1/N}^{1} |f - 1| .$$

Since $J_3 \to 0$ as $n \to \infty$, it follows that $\int_{1/N}^{1} |f - 1| = 0$, that is, $f(x) = 1$ for $x \in [1/N, 1]$. Since N is arbitrary, it follows that $f(x) = 1$ for $x \in (0, 1]$. Thus if $f = \lim f_n$ in the NLS, then

$$f(x) = \begin{cases} 0 & \text{if } x \in [-1, 0] \\ 1 & \text{if } x \in (0, 1]. \end{cases}$$

Thus f cannot be continuous. This contradictions shows that the Cauchy sequence (f_n) is not convergent in $(C[-1, 1], \| \ \|_1)$. ☐

Note that this along with Items 28c answers the question raised in Item 20 in the negative. One can still give a direct negative answer to the question!

Items 32–39 were done on 29-07-2011.

40. The normed linear space $(\ell^1, \| \ \|_1)$ is complete.

We discussed it in the class. We used the curry-leaves trick to avoid the ubiquitous argument: Keeping n fixed and letting $m \to \infty$. Learn the proof from your classmates.

Let $z_n \in \ell^1$. Write it as $z_n = (z_{nk})_{k=1}^{\infty}$. Assume that (z_n) is Cauchy in ℓ^1. We need to find a $z \in \ell^1$ such that $\| z_n - z \|_1 \to 0$.

Write this as an infinite rectangular array.

$$\begin{array}{rcl} z_1 & = & z_{11}, z_{12}, z_{13}, \ldots, z_{1k}, \ldots, \\ z_2 & = & z_{21}, z_{22}, z_{23}, \ldots, z_{2k}, \ldots, \\ & \vdots & \\ z_n & = & z_{n1}, z_{n2}, z_{n3}, \ldots, z_{nk}, \ldots, \\ & \vdots & \end{array}$$

Fix $k \in \mathbb{N}$. We claim that the k-th column, considered as a sequence $(z_{nk})_{n=1}^{\infty}$ is Cauchy in $\mathbb{K}$:

$$|z_{nk} - z_{mk}| \leq \sum_k |z_{nk} - z_{mk}| = \| z_n - z_m \|_1 \to 0,$$

as $n, m \to \infty$. Since $\mathbb{K}$ is complete, this sequence converges, say, to α_k, that is, $\lim_n z_{nk} = \alpha_k$. This being true for each $k \in \mathbb{N}$, we obtain a sequence $\alpha = (\alpha_k)$. Our aim is to show that (1) $\alpha \in \ell^1$ and that (2) $\| z_n - \alpha \|_1 \to 0$.

To prove (1), we need to show that the series $\sum_k |\alpha_k|$ is convergent, that is to show that the sequence $(\sum_{k=1}^{N} |\alpha_k|)$ of its partial sums is bounded above. This means that we need to estimate $\sum_{k=1}^N |\alpha_k|$ in a 'uniform' way so that the upper bound is independent of N. The only fact we have at our disposal is that $z_{nk} \to \alpha_k$ for each k. We have thus arrived at an obvious estimate:

$$\begin{aligned}\sum_{k=1}^N |\alpha_k| &\leq \sum_{k=1}^N |\alpha_k - z_{nk}| + \sum_{k=1}^N |z_{nk}| \\ &\leq \sum_{k=1}^N |\alpha_k - z_{nk}| + \sum_{k=1}^\infty |z_{nk}| \\ &= \sum_{k=1}^N |\alpha_k - z_{nk}| + \|z_n\|_1 .\end{aligned}$$

Since (z_n) is a Cauchy in ℓ^1, it is bounded and hence there exists $M > 0$ such that $\|z_n\|_1 \leq M$ for each n. This entails the following estimate:

$$\sum_{k=1}^N |\alpha_k| \leq \sum_{k=1}^N |\alpha_k - z_{nk}| + M.$$

Thus we reduced our problem of finding a 'uniform' estimate for $\sum_{k=1}^N |\alpha_k|$ to that of finding one such for the sum $\sum_{k=1}^N |\alpha_k - z_{nk}|$. Since for each fixed k, $1 \leq k \leq N$, we know that $z_{nk} \to \alpha_k$, given $\varepsilon = 1$, say, there exists $n_0(k) \in \mathbb{N}$ such that

$$n \geq n_0(k) \implies |\alpha_k - z_{nk}| \leq 1/N, \text{ for } 1 \leq k \leq N.$$

If we let $n_0 := \max\{n_0(k) : 1 \leq k \leq N\}$, then it follows that

$$n \geq n_0 \implies \sum_{k=1}^N |\alpha_k| \leq 1 + M.$$

Hence (1) is established.

We now prove (2). Again this means that we need to estimate $\sum_{k=1}^N |\alpha_k - z_{nk}|$ uniformly in N. We use the curry-leaves trick here. Review the proof in Item 43. Observe

$$\begin{aligned}\sum_{k=1}^N |\alpha_k - z_{nk}| &\leq \sum_{k=1}^N |\alpha_k - z_{mk}| + \sum_{k=1}^N |\alpha_{mk} - z_{nk}| \quad \text{(for any } m\text{)} \\ &\leq \sum_{k=1}^N |\alpha_k - z_{mk}| + \|z_m - z_n\| \quad \text{(for any } m\text{)} .\end{aligned}$$

Let $\varepsilon > 0$ be given. Using Cauchy nature of (z_n), there exists n_1 such that for $n, m \geq n_1$, we have $\|z_m - z_n\| < \varepsilon/2$. We learnt how to estimate the first term on the right side during our discussion on (1). Choose n_0 such that for

$$\text{for } 1 \leq k \leq N \text{ and for } m \geq n_0, \text{ we have } |\alpha_k - z_{mk}| < \varepsilon/2N$$

so that $\sum_{k=1}^N |\alpha_k - z_{mk}| < \varepsilon/2$ for any $m \geq n_0$. Consequently, we arrive at, for $n \geq n_1$

$$\begin{aligned}\sum_{k=1}^N |\alpha_k - z_{nk}| &\leq \sum_{k=1}^N |\alpha_k - z_{mk}| + \|z_m - z_n\| \\ &< \varepsilon/2 + \varepsilon/2 \text{ for all } m \geq \max\{n_0, n_1\}.\end{aligned}$$

Since this is true for all N, we get the desired result $\| \alpha - z_n \| < \varepsilon$ for $n \geq n_1$. □

41. The same proof establishes the completeness of the spaces $(\ell^p, \| \ \|_p)$ for $1 \leq p < \infty$. We urge the reader to work out the details as it will help them reinforce the understanding of the material in the last item.

    The space $(\ell^\infty, \| \ \|_\infty)$ is a special case of Item 37.

42. We reviewed basic concepts/results from Measure theory such as simple functions, their integrals, integrals of non-negative measurable functions, MCT, integrable functions and the linearity of the integrals, DCT and Fatou's lemma.

    Items 40–42 were done on 02-08-2011.

43. We wish to introduce the Lebesgue spaces $L^p(X, \mathcal{B}, \mu)$, $1 \leq p \leq \infty$. . For this purpose we need the integral versions of Hölder's and Minkowski's inequalities.

44. Let $(X, \mathcal{B}, \mu)$ be a measure space and $1 \leq p < \infty$. Let q be the conjugate index defined by the equation $\frac{1}{p} + \frac{1}{q} = 1$. Let $\| f \|_p := (\int_X |f|^p)^{1/p}$ and $\| g \|_q := (\int_X |g|^q)^{1/q}$ for all measurable functions $f, g \colon \to \mathbb{C}$.

    Let $x := \frac{|f(t)|}{\| f \|_p}$ and $y := \frac{|g(t)|}{\| g \|_q}$ in Young's inequality and integrate. We then obtain the integral version of Hölder's inequality:

$$\int_X |fg| d\mu \leq \| f \|_p \| g \|_q . \tag{8}$$

    The equality case of Young's inequality leads us to conclude that equality occurs in (8) iff there exist constants α and β, not both zero, such that $\alpha |f|^p = \beta |g|^q$.

45. We now prove the integral version of Minkowski's inequality: $\| f + g \|_p \leq \| f \|_p + \| g \|_p$ for measurable functions f and g such that $\int_X |f|^p d\mu < \infty$ and $\int_X |g|^p d\mu < \infty$. *Hint:* Start with

$$|f + g|^p \leq |f||f + g|^{p-1} + |g||f + g|^{p-1}.$$

    and proceed as in Item 24.

46. Let $(X, \mathcal{B}, \mu)$ be a measure space. Let

$$L^p(X, \mathcal{B}, \mu) := \{f \colon X \to \mathbb{C} : f \text{ is measurable and } \int_X |f|^p d\mu < \infty\}.$$

    Then L^p is a linear space over $\mathbb{C}$. The map $f \mapsto \| f \|_p$ is only a semi-norm in the sense that $\| f \|_p = 0$ does not imply that $f = 0$. However, it implies that $f = 0$ a.e. Hence if we agree to identify two measurable functions f, g if $f = g$ a.e., then $L^p(X, \mathcal{B}, \mu)$ becomes a normed linear space.

    If we say $f \in L^p(\mathbb{R})$ is continuous, what we mean is the existence of a continuous function g such that $f = g$ a.e. Necessarily, $\| g \|_p < \infty$!

    If you are more pedantic, then we need to introduce the notion of a seminorm p on a vector space X, talk of the equivalence relation $x_1 \sim x_1$ iff $p(x_1 - x_2) = 0$, form the quotient $X/\sim$ and introduce a norm on the equivalence classes $\| [x] \| := p(x)$ and so on. I believe that it is easier to work with L^p-spaces the way we explained above.

47. **(Riesz-Fischer Theorem)** Let $(X, \mathcal{B}, \mu)$ be a measure space and $1 \le p < \infty$. Then $(L^p(X), \| \ \|_p)$ is complete.

For a proof, refer to any standard book such as Rudin's *Real and Complex Analysis*, Page 69 or [A-B], Page 258.

48. An important corollary of the proof is the following:
If $f_n \to f$ in L^p, then there exists a subsequence (f_{n_k}) such that $f_{n_k}(x) \to f(x)$ almost everywhere. This technical observation will be very useful often when dealing with convergence in L^p.

We also gave examples to show that cannot conclude that f_n converges pointwise to f a.e. (See Item 54.)

49. We now deal with L^∞. The obvious choice of $\sup\{|f(x)|\}$ is bad, since we want identify functions which agree a.e. For instance, if $f(x) = 1$ for $x \in \mathbb{R}$ and $g(x) = \begin{cases} 1 & \text{if } x \notin \mathbb{Q} \\ x & \text{otherwise} \end{cases}$, then 1 is a bound for f but g is unbounded. But $f = g$ a.e. under Lebesgue measure. Hence we need to think smart.

Let $(X, \mathcal{B}, \mu)$ be a measure space and f be measurable. A real number α is said to be an essential upper bound for f if $f(x) \le \alpha$ a.e. and f is said to be essentially bounded. If f has an essential upper bound then it has a least one, called essential supremum of f and denoted by $\| f \|_\infty$.

Do NOT confuse this norm the sup norm! For instance, the function f defined above is essentially bounded with 1 as the essential supremum though it is unbounded with ∞ as its supremum.

Let L^∞ denote the set of all essentially bounded measurable functions on X. Then it is a complete normed linear space with respect to the norm $f \mapsto \| f \|_\infty$. This is a good exercise. The crucial observation here is that if $f_n \to f$ in L^∞, then the convergence is uniform on X minus a set of measure 0. For, given $\varepsilon > 0$, there exists N_ε such that for $n \ge N_\varepsilon$, we have $\| f - f_n \|_\infty < \varepsilon$. This means that there exists E_n with $\mu(E_n) = 0$ and $|f(x) - f_n(x)| < \varepsilon$ for $x \in X \setminus E_n$, $n \ge N$. Let $A_\varepsilon := \cup_{n \ge N_\varepsilon} E_n$. Then $\mu(E) = 0$ and we have Check this carefully!

$$n \ge N, x \in X \setminus A_\varepsilon \implies |f(x) - f_n(x)| < \varepsilon.$$

Let $A := \cup_k A_{1/k}$. Then $\mu(A) = 0$ and we have for any $k \in \mathbb{N}$ and the corresponding $N_k := N_{1/k}$

$$n \ge N_k \text{ and } x \in X \setminus A \implies |f(x) - f_n(x)| < 1/k.$$

With these ideas, now you can proceed on your own to establish the completeness of L^∞.

An important observation which will be used often when dealing with L^∞:

$$\text{If } f \in L^\infty \text{ and } \alpha < \| f \|_\infty, \text{ then } \mu(\{x \in X : |f(x)| > \alpha\}) > 0. \tag{9}$$

50. The set of simple functions in $L^p(X, \mathcal{B}, \mu)$ is dense in L^p for $1 \le p < \infty$. You reduce the problem to that of finding a simple function s which ε-close to a given non-negative measurable function f. But this follows from the way $\int f \, d\mu$ is defined!

Items 43–50 were done on 03-08-2011.

51. Let (x_n) be a sequence in a normed linear space X. Let $s_n := \sum_1^n x_k$ be the nth partial sum. If the sequence (s_n) converges to an $x \in X$, we then say $\sum_1^\infty x_k$ is convergent and call x the sum of the infinite series $\sum_1^\infty x_k$. We write this as $x = \sum_1^\infty x_k$.

    A series $\sum_1^\infty x_n$ in a normed linear space X is *absolutely convergent* or *norm convergent* if $\sum_1^\infty \|x_n\|$ is convergent.

52. The proof of the completeness of L^p leads us to the following result which may be considered as an abstract M-test.

53. A normed linear space X is Banach iff every absolutely convergent series is convergent.

    If (x_n) is Cauchy in X, choose n_k such that $\|x_n - x_m\| < 2^{-k}$ for all $m, n \geq n_k$. Define $y_1 := x_{n_1}$, $y_k := x_{n_k} - x_{n_{k-1}}$ for $k \geq 2$. Then $y_1 + \cdots + y_k := x_{n_k}$. Note that the series $\sum_k y_k$ is absolutely convergent. Elaborate on this

54. **Convergence in L^p.** Beginners usually have problems with this notion. This set of exercises deals with relations between pointwise convergence and convergence in L^p-norm.

    (a) $f_n \to f$ in L^p but f_n may not converge pointwise (a.e.) to f. Consider $L^p[0,1]$. Let $J = [j - 1/k, j/k]$, $1 \leq j \leq k$, $k \in \mathbb{N}$. Let χ_J be its characteristic function. Then the collection $\{\chi_J\}$ is countable and hence be written as a sequence, say, (f_n). Then $\|f_n\|_p \to 0$. But f_n does not converge to 0 pointwise. (Note that this is true no matter how the χ_J's are enumerated as a sequence.)

    (b) $f_n \in L^p$ may converge pointwise to some f and f may lie in L^p. But f_n may not converge to f in L^p. Again, consider $L^p[0,1]$. Let $f_n := n\chi_{[0,1/n]}$. Then f_n converges pointwise to 0 a.e but does not converge to 0 in L^p.

    (c) Let $f_n, f \in L^p$. Assume that $\|f_n - f\|_p \to 0$ and that $\sum_n \|f_n - f\|$ is convergent. Then $f_n \to f$ pointwise a.e.

55. Exercise. Let Y be a vector subspace of an NLS X. Show that the closure $\overline{Y}$ of Y is also a vector subspace.

56. **Convex Sets.** Let V be a real vector space. A subset $A \subset V$ is said to be *convex* if for $x, y \in A$, and $0 \leq t \leq 1$, we have $tx + (1-t)y \in A$. The following are immediate.

    (a) Any vector subspace W of V is convex.

    (b) Intersection of convex subsets is convex.

    (c) Given a subset A, there exists a smallest convex set containing A, called the *convex hull* of A, denoted by $CH(A)$.

    Compare this with the notions of (1) the smallest subgroup containing A, a subset of a group G, (2) the smallest ideal containing A in a ring R, (3) the smallest vector subspace containing A, a subset of a vector space V. In each of these cases, there is a description of the smallest object.

    (d) How to describe $CH(A)$?

    (e) What is the convex hull of the unit circle in $\mathbb{R}^2$?

    (f) Let A and B be convex subsets of a vector space. Is $A + B$ convex? Is λA convex where λ is a scalar?

(g) (Exercise.) Given a convex set C and positive scalars λ and μ, show that $\lambda A + \mu A = (\lambda + \mu)A$.

(h) Let X be a NLS. Then $B(a, r)$ is a convex subset of X.

(i) Let X be a NLS and C be a convex subset. Then $\overline{C}$ is convex.

(j) (Exercise.) Let ClCH(A) denote the smallest closed convex subset containing A in an NLS X. Is it true that ClCH(A) is the same as the $\overline{CH(A)}$?

(k) (Exercise.) Let K be a compact subset of a Banach space X. Show that ClCH(K) is compact.

(l) (Exercise.) Let (x_n) be a sequence in a Banach space. Assume that $x_n \to 0$. Let $A := \{x_n : n \in \mathbb{N}\}$. Show that CLCH($A$) is compact and it consists of elements of the form $\sum_1^\infty t_n x_n$, where $t_n \geq 0$ and $\sum_n t_n = 1$. (This uses notions to be introduced later. So it should be shifted to appropriate place.)

57. Exercise. Let A and B be compact subsets of an NLS X. Show that $A + B$ is compact.

58. Recall Weierstrass approximation theorem and the existence of continuous but not differentiable functions. These lead us to conclude that the space $C^1[0, 1]$ is not closed in $(C[0, 1], \| \ \|_\infty)$. In particular, $(C^1[0, 1], \| \ \|_\infty)$ is not complete.

59. In view of the last item, we wish to endow $C^1[0, 1]$ with a norm which will ensure that if f_n is Cauchy in the norm, then it converges to a C^1-function. Since $\| \ \|_\infty$ ensures that a Cauchy sequence (f_n) in the norm $\| \ \|_\infty$ converges to a continuous function, we conclude that our new norm should be 'stronger' than $\| \ \|_\infty$.

    If you recall a standard result learnt in Uniform Convergence (See Theorem 8.2.3 of Bartle-Sherbert, Page 235), we shall see that we need to ensure that the sequence should converge at some point x_0 and (f_n') should be uniformly Cauchy. All these considerations lead us to at least three norms.

    Norms which ensure that f_n converges in $C^1[0, 1]$ iff both (f_n) and (f_n') are uniformly convergent are
    (1) $\|f\|_{C^1} := \|f\|_\infty + \|f'\|_\infty$.
    (2) $\|f\|_{C^1} := |f(0)| + \|f'\|_\infty$
    (3) $\|f\|_{C^1} := \max\{\|f\|_\infty, \|f'\|_\infty\}$.

    It is easy to see that these norms are equivalent.

    $(C^1[0, 1], \| \ \|_{C^1})$ is complete. Use theorems of real analysis and/or fundamental theorems of integral calculus.

    Items 51–59 were done on 04-08-2011.

    The next few items introduces NLS's of holomorphic/analytic functions.

60. We need the following results from complex analysis.

    (a) Gauss Mean Value Theorem for holomorphic functions. See Theorem 6.12 of Bak & Newman/page 76.

    (b) Morera's theorem or rather its consequence, Weierstrass theorem which states that if a sequence of holomorphic functions (defined on an open set U) converge uniformly on compact subsets of U, then the limit function is holomorphic. See Theorem 7.6 of Bak & Newman/page 87.

61. Let $U \subset \mathbb{C}$ be a nonempty open set. Let $B_H(U)$ denote the vector subspace of $B(U, \mathbb{C})$ consisting of holomorphic/analytic functions on U. Then $B_H(U)$ is complete under $\| \|_\infty$. For if (f_n) is Cauchy in $B_H(U)$, then it is uniformly Cauchy on U and hence it converges to a continuous function on U. By Morera/Weierstrass theorem, it follows that f is holomorphic on U.

62. **(Bergman Space)** Let $\mathbb{D} := \{z \in \mathbb{C} : |z| < 1\}$ be the open unit disk in $\mathbb{C}$. Let $L^2_H(\mathbb{D}, dA)$ be the space of all holomorphic functions on the unit disk which are in $L^2(\mathbb{D})$ with respect to the standard Lebesgue (area) measure. In terms of Cartesian coordinates $dA = dxdy$ and in terms of polar coordinates $dA = r\,drd\theta$. We claim that $L^2_H(\mathbb{D}, dA)$ is a Hilbert space with the inner product inherited from $L^2(\mathbb{D})$.

    This proof is instructive for budding analysts. If we are given a Cauchy sequence in L^2-norm, we want to estimate pointwise $|f_n(z) - f_m(z)|$ perhaps in a uniform way so that we can appeal to Weierstrass theorem of Item 60b. Thus we need to look for an integral representation of $f(z)$ and use it to get a pointwise estimate. This is provided by the mean value theorem. But as we work in L^2, we apply Cauchy-Schwarz to get a pointwise estimate in terms of L^2-norms.

    The precise proof follows from the following steps.

    (a) **"Solid" Mean Value Theorem**. Let $f \in H(B(a, R))$. Then for any $0 < r < R$, we have
    $$f(a) = \frac{1}{\pi r^2} \int_{B(a,r)} f\, dA. \tag{10}$$
    Start with the standard (Gauss) mean value theorem and multiply it by any $0 < s < r$ and integrate with respect to s to obtain (10).

    (b) Let $z_0 \in \mathbb{D}$. Then for any $f \in L^2_H(\mathbb{D}, dA)$, we have

    A factor of $\sqrt{\pi}$ may be missing!
    $$|f(z_0)| \leq \frac{1}{1 - |z_0|} \|f\|_2 \,. \tag{11}$$

    (c) $L^2_H(\mathbb{D}, dA)$ is complete. We need only prove that $L^2_H(\mathbb{D}, dA)$ is a closed subspace of $L^2(\mathbb{D})$. Let $f_n \in L^2_H(\mathbb{D}, dA)$ converge to some $f \in L^2(\mathbb{D})$. We know that there exists a subsequence of (f_n) which converges pointwise to f almost everywhere. So, WLOG, assume that $f_n \to f$ pointwise a.e. Note that, for $|z| < r < 1$, it follows from (11) that $|f_m(z) - f_n(z)| \leq \frac{1}{1-r} \|f_m - f_n\|_2$. Thus (f_n) is uniformly Cauchy on subdisks of $\mathbb{D}$. Hence by Weierstrass theorem (or by Morera's theorem), f_n converges pointwise (in fact, uniformly on subdisks) to a holomorphic function g, say. Hence $f = g$ a.e.

    Go through the proof. Observe how we got around to pointwise (in fact uniform on compact subsets) estimate of a holomorphic function in terms of its L^2 estimate.

63. Show that the subset $C[a, b]$ of continuous functions in $L^p[a, b]$ is dense for $1 \leq p < \infty$. *Hint:* Enough to approximate χ_E by a continuous function. Recall regularity of Lebesgue measure and Urysohn's lemma.

Items 60–63 were done on 05-08-2011.

64. **Exercises.** The following set of exercises was given on 28/08/2011. Written solutions are to be submitted on 16/08/2011.

  (a) Let X, Y be normed linear spaces. If $T\colon X \to Y$ is continuous at $x_0 \in X$ show that T is uniformly continuous on X.

  (b) Let X be a normed linear space. Fix $v \in X$ and $0 \neq \lambda \in \mathbb{K}$. Show that the translation $T_v : x \mapsto x + v$ and the homothety $x \mapsto \lambda x$ are homeomorphisms.

  (c) Let $U, V \subset X$ be open subsets of a normed linear space X. Show that $U + V$ is open.

  (d) Let V be a vector subspace of a normed linear space X. Assume that the interior of V is nonempty. What can you conclude about V?

  (e) Let $\mathbf{c}_{00} := \{z \in \ell^1 : \exists N = N(z) \text{ such that } x_n = 0 \text{ for } n \geq N\}$. Show that $\mathbf{c}_{00}$ is dense in ℓ^p for $1 \leq p < \infty$. Is it dense in ℓ^∞?

  (f) Let $\mathbf{c}$ be the set of all convergent sequences in ℓ^∞. Is it closed in ℓ^∞?

  (g) Let $\mathbf{c}_0 \subset \ell^\infty$ be the set of all sequences converging to 0. Is $\mathbf{c}_0$ closed in ℓ^∞? in $\mathbf{c}$?

  (h) True or false? V is a proper vector subspace of a normed linear space X. Then V is nowhere dense in X.

  (i) Let X and Y be normed linear spaces. For $(x, y) \in X \oplus Y$, define $\|(x, y)\| := \max\{\|x\|, \|y\|\}$. Then $(x, y) \mapsto \|(x, y)\|$ is a norm on $X \oplus Y$, called a product norm on the direct sum.
  (i) Show that the topology induced by this norm is the product topology on $X \oplus Y$. (Note that the underlying set of $X \oplus Y$ is $X \times Y$!)
  (ii) Show that $(x, y) \mapsto \|x\| + \|y\|$ is a norm on $X \oplus Y$ equivalent to the earlier one.

  (j) Show that the vector addition and the scalar multiplication are continuous. That is, the map $(x, y) \mapsto x + y$ from $X \oplus X \to X$ and the map $(\lambda, x) \mapsto \lambda x$ from $\mathbb{K} \times X \to X$ are continuous.

  (k) Let the notation be as in Ex. 64i. Consider the projection maps $\pi_X\colon X \oplus Y \to X$ defined by $\pi_X(x, y) = x$. Is π_X continuous?

  (l) Let $x_n \to x$ in a normed linear space X. Show that $a_n := (x_1 + \cdots + x_n)/n$ converges to x in X.

  (m) Let X be a normed linear space. Let $K \subset X$ be compact and $C \subset X$ be closed. Show that $K + C$ is closed in X.

  (n) Consider $\Lambda\colon \mathbf{c} \to \mathbb{C}$ be defined by $\Lambda(z) := \lim_n z_n$. Is Λ continuous on $\mathbf{c}$?

65. Look at the examples of Banach spaces. Which of them are infinite dimensional?

  Look at the examples of infinite dimensional Banach Spaces. Are there Banach spaces whose dimension is countably infinite? The next explains their non-existence.

66. Let X be a Banach space. Assume that $\dim X = \infty$. Then $\dim_{\mathbb{K}} X$ is uncountable. Assume the contrary. Let $\{e_n : n \in \mathbb{N}\}$ be a countable basis of X. Let $Y_n := \operatorname{span}\{e_1, \ldots, e_n\}$. Then Y_n are proper, closed and nowhere dense. Their union is X. Recall Baire.

67. Let $(V, \langle\ ,\ \rangle)$ be a finite dimensional inner product space. Let W be a proper (necessarily closed) vector subspace of V. Then there exists a unit vector $u \in V$ such that $d(u, W) = 1$.

    In an arbitrary (infinite dimensional) normed linear space this may not be true. See Item 71. The best we have is Riesz lemma.

68. **(Riesz Lemma)** Let X be any normed linear space. Let Y be a proper closed vector subspace of X. Let $0 < \varepsilon < 1$ be given. The there exists $u \in X$ such that $\|u\| = 1$ and $d(u, Y) \geq \varepsilon$.

    We prove this for $\varepsilon = 1/2$. Choose any $x \in X \setminus Y$. Then $\delta := d(x, Y) > 0$. Let $y_0 \in Y$ be such that $\delta \leq d(x, y_0) < 2\delta$. Then $u := \frac{x - y_0}{\|x - y_0\|}$ will do the job. (Exercise: Modify this to prove the stated result.)

69. A topological space X is said to be *locally compact* iff for any $x \in X$ and an open set $U \ni x$ there exists an open set V with compact closure such that $x \in V \subset \overline{V} \subset U$.

    A normed linear space X is locally compact iff $B[0, 1]$ is compact.

70. A normed linear space X is locally compact iff it is finite dimensional.

    Assume that X is locally compact but not f.d. Let u_1 be a any unit vector. Choose by induction a unit vector $u_n \notin Y_{n-1} := \text{span}\,\{u_1, \ldots, u_{n-1}\}$ such that $d(u_n, Y_{n-1}) \geq 1/2$. Then (u_n) is a sequence in the compact set $B[0, 1]$ with no convergent subsequence.

71. The last result is used as follows. One finds the space of solutions of an equation and to show that the space of solutions is finite dimensional, we show that it is a locally compact normed linear space.

72. Let $X := \{f \in C[0, 1] : f(0) = 0\}$ with the norm $\|\ \|_\infty$. Consider $Y := \{f \in X : \int_0^1 f = 0\}$. Then there does not exists a $u \in X$ with $\|u\|_\infty = 1$ and $d(u, Y) = 1$. (Exercise)

Items 65–72 were done on 05-08-2011.

73. **Exercise.** Let $\text{Lip}([a, b], \mathbb{R})$ be the set of all Lipschitz functions from $[a, b]$ to $\mathbb{R}$. Show that it a vector space under obvious operations. Think of a norm on this space to make it Banach. *Hint:* Recall what we did in the case of C^1-functions.

74. We reviewed the concept of quotients in Linear Algebra. Looked at some geometric example to motivate the next item

75. Let X be a normed linear space. Let Y be a vector subspace. Let $\tilde{x}$ stand for the coset $x + Y$. Geometry motivated us to define $\|\tilde{x}\| := d(x, Y)$. We verified this is well-defined. If we want to make sure that this is a norm, we saw that we needed to assume that Y is closed. If Y is closed, then $\tilde{x} \mapsto d(x, Y)$ is a norm on the quotient X/Y, called the quotient norm. Observe that $\|\tilde{x}\| \leq \|x\|$.

76. If the quotient X/Y is finite dimensional, then it is complete whether or not X is complete.

77. Let X be Banach and $Y \leq X$ be closed. Then X/Y with quotient norm is complete. Enough to show that if $\sum_n \|\tilde{x}_n\| < \infty$, then $\sum_n \tilde{x_n}$ is convergent. Choose $t_n \in \tilde{x_n}$ such

that $\|t_n\| \leq \|\tilde{x}_n\| + 2^{-n}$. Then $\sum_n t_n$ converges, say to $t \in X$. Observe that

$$\left\|\tilde{t} - \sum_{k=1}^{n} \tilde{x}_n\right\| = \left\|\tilde{t} - \sum_{k=1}^{n} \tilde{t}_n\right\| \leq \left\|t - \sum_{k=1}^{n} t_n\right\| \to 0,$$

as $n \to \infty$.

78. Note that the quotient map $\pi\colon X \to X/Y$ is continuous.

79. As an application, let us prove the following. Let X be a normed linear space. Let Y be closed and V be a finite dimensional vector subspaces of X. Then $V + Y$ is closed. *Hint.* Observe that $V + Y = \pi^{-1}(\pi(V))$. See Item 90.

80. In general, the above result is not true if we assume Y and Y are merely closed. We give an example. The details are left to the reader as an exercise.

    Consider the following subsets of ℓ^2.

$$\begin{aligned} V &= \{x \in \ell^2 : x_{2n} = 0, \forall n \in \mathbb{N}\} \\ W &= \{x \in \ell^2 : x_{2n} = \frac{1}{2n} x_{2n-1}, \forall n \in \mathbb{N}\}. \end{aligned}$$

    Then V and W are closed vector subspaces of ℓ^2 but $V + W$ is not closed. *Hint:* Show that the closure of $V + W$ is ℓ^2. Consider the sequences $v_n = e_1 + e_3 + \cdots + e_{2n-1}$ and $w_n = v_n + 2^{-1}e_2 + 4^{-1}e_4 + \cdots + (2n)^{-1}e_{2n}$.

    Can you adapt this argument to produce such subspaces in ℓ^1?

81. **Exercise.** Let X denote the set of all "double sequences" of the form

$$(b, a) := (\ldots, b_{-3}, b_{-2}, b_{-1}, a_0, a_1, a_2, \ldots)$$

    such that $\lim_n a_n$ exists and $\lim_n b_{-n}$ exists. Let $\|(b, a)\| := \sup\{|a_j|, |b_{-k}|\}$. Then X is a Banach space with this norm. The space Y of (b, a) such that the limits of the a and b-sequences are 0 is closed. What is X/Y? What is the quotient norm? *Hint.* I'd like to start with the easy question of 'single sequences'.

We now introduce Hilbert spaces — complete inner product spaces and establish some important properties.

82. Let X be a vector space and $\langle\ ,\ \rangle$ be an inner product on X. Let $\|x\| := (\langle x, x\rangle)^{1/2}$. Recall the crucial Cauchy-Schwarz inequality:

$$|\langle x, y\rangle| \leq \|x\| \, \|y\| \text{ for all } x, y \in X. \tag{12}$$

    To prove this, it is enough to show that $|\langle x, y\rangle|^2 \leq \langle x, x\rangle \langle y, y\rangle$. If we know t^2 for all real numbers t, to get the 'mixed term' like st, our high-school algebra suggests to look $(s+t)^2$ and $(s-t)^2$. Since we are dealing with a vector space, we look at $\langle x + \lambda y, x + \lambda y\rangle$ where $\lambda \in \mathbb{K}$. We obtain

$$\langle x + \lambda y, x + \lambda y\rangle = \langle x, x\rangle + \overline{\lambda} \langle x, y\rangle + \lambda\overline{\langle x, y\rangle} + |\lambda|^2 \langle y, y\rangle.$$

Note that the middle two terms are conjugates of each other. Since we need to estimate $|\langle x, y\rangle|^2$, an obvious choice for λ is something like $\pm\langle x, y\rangle$. If we look at the fourth term, after a little thought we arrive at $\lambda := -\frac{\langle x,y\rangle}{\langle y,y\rangle}$. Using this choice of λ in $\langle x + \lambda y, x + \lambda y\rangle \geq 0$ and simplifying we get the required inequality.

The proof above shows when the equality holds in the C-S inequality.

Items 73–82 were done on 09-08-2011.

83. Let X be an inner product space. We say that $x, y \in X$ are *orthogonal* to each other if $\langle x, y\rangle = 0$. The following are easy to check.

    (a) If x and y are orthogonal to each other then $\|x + y\|^2 = \|x\|^2 + \|y\|^2$. (Pythagoras Theorem.)

    (b) Given a set A of vectors such that any two distinct elements of A are orthogonal (to each other), then we say A is an *orthogonal* set.

    Any orthogonal set (consisting of **non-zero** vectors) is easily seen to be linearly independent.

    If $A = \{x_k : 1 \leq k \leq n\}$ is an orthogonal set, it follows from Pythagoras theorem (by induction) that
    $$\left(\left\|\sum_{k=1}^{n} x_k\right\|\right)^2 = \sum_{k=1}^{n} \|x_k\|^2 .$$

    (c) Fix $y \in X$. Then the map $f_y(x) := \langle x, y\rangle$ is $\mathbb{K}$-linear and continuous: $|f_y(x)| \leq \|x\| \|y\|$. Hence its kernel is a closed vector subspace of X.

    (d) Let $S \subset X$. We let $S^\perp := \{x \in X : \langle x, s\rangle = 0 \text{ for all } s \in S\}$. Then $S^\perp$ is a closed vector subspace.

    (e) **Parallelogram identity.** For any $x, y \in X$ one has
    $$\|x + y\|^2 + \|x - y\|^2 = 2\left(\|x\|^2 + \|y\|^2\right). \tag{13}$$

    (f) If $\{x_k | 1 \leq k \leq n\}$ is a linearly independent set, then the Gram-Schmidt process yields a set $\{e_k : 1 \leq k \leq n\}$ with the property that $\langle e_j, e_k\rangle = \delta_{ij}$. The set $\{e_k : 1 \leq k \leq n\}$ is called an *orthonormal* set.

84. Review of orthogonal decomposition in FD-IPS. Given a vector subspace W of a finite dimensional inner product space V, we have an orthogonal decomposition $V = W \oplus W^\perp$. From this one deduces the following: If $x \in X$ then there exists a unique $w \in W$ such that $\|x - w\| \leq \|x - y\|$ for all $y \in W$.

    We prove such a result in Hilbert spaces under the assumption that the subspace is closed. Look at the next item to see why we need Y to be closed.

85. **(Exercise.)** Let $X = \ell^2$. Let Y be the subspace of all elements x in X with the property that all but finitely many x_n's are zero. What is $Y^\perp$?

    If $v = (1, 1/2, 1/3, \ldots, 1/n, \ldots)$, does there exist a $y_0 \in Y$ such that $\|v - y_0\| \leq \|v - y\|$ for $y \in Y$? What is $d(v, Y)$?

86.

87. Let C be a (non-empty) closed convex subset of a Hilbert space H. Then there a unique $a \in C$ such that $\|a\| \leq \|x\|$ for all $x \in C$.

Idea: Let $\delta := \inf\{\|x\| : x \in C\}$. Then there exists $x_n \in C$ such that $\|x_n\| \to \delta$. If we can show that x_n converges to an $a \in X$, then $a \in C$ and we are done. Since H is complete, enough to show that (x_n) is Cauchy. That is, we need to estimate $\|x_n - x_m\|$. In an inner product space, it is wise to estimate $\|x_n - x_m\|^2$. How to do this? Parallelogram identity (13) does it for us.

$$\|x_n - x_m\|^2 = 2\|x_n\|^2 + 2\|x_m\|^2 - \|x_n + x_m\|^2 . \tag{14}$$

Note that the first two terms on RHS go to $4\delta^2$. We rewrite the third term on the RHS to exploit convexity of C.

$$\|x_n + x_m\|^2 = \left\|2\left(\frac{1}{2}x_n + \frac{1}{2}x_m\right)\right\|^2 = 4\left\|\left(\frac{1}{2}x_n + \frac{1}{2}x_m\right)\right\|^2 \geq 4\delta^2, \tag{15}$$

since $(\frac{1}{2}x_n + \frac{1}{2}x_m) \in C$. Now complete the proof of existence of a.

If $a, b \in C$ are two such, then use (14) replacing x_m and x_n with a and b. Argue as in (15). Deduce $a = b$.

88. Let W be a closed vector subspace of a Hilbert space H. Let $x \in H$. Observe that $C := x + W$ is a closed convex set. If $a \in C$ is of smallest norm, then $a = x - y_0$ with $y_0 \in W$. Hence

$$\|x - y_0\| = d(x, y_0) \leq d(x, W) \leq \|x - y\| \text{ for } y \in W.$$

89. Let X be a normed linear space. Let Y be a proper closed subspace and $x \in X \setminus Y$. Let $z = y + \lambda x$ with $y \in Y$ and $\lambda \in \mathbb{K}$. Then there exists $C > 0$ such that $|\lambda| \leq C\|z\|$.

Intuitively, this is to be expected. For consider the linear map from normed linear space $Y + \mathbb{K}x \to \mathbb{K}$ given by $y + \lambda x \to \lambda$. We 'expect' it to be continuous.

Idea: Let $\delta := d(x, Y)$. Hence $\left\|x - \frac{-1}{\lambda}y\right\| \geq \delta$. That is, $\left\|\frac{1}{\lambda}z\right\| \geq \delta$.

90. The last item yields the following. If Y is closed vector subspace of X and $x \in X$, then $Y + \mathbb{K}x$ is closed in Y.

If $x \in Y$, nothing to prove. So, assume $x \notin Y$. If $y_n + \lambda_n x \to v$. We obtain from the last item,

$$|\lambda_n - \lambda_m| \leq C\|(y_n + \lambda_x) - (y_m + \lambda_m x)\| \to 0.$$

Hence $\lambda_n \to \lambda$ for some $\lambda \in \mathbb{K}$. Hence $y_n = v - \lambda_n x \to v - \lambda x$. Since Y is closed it follows that $v - \lambda x = y_0 \in Y$, say. Thus, $v = y_0 + \lambda x$.

By induction the result of Item 79 follows.

Items 83–88 were done on 10-08-2011.

91. Let the notation be as in Item 88. We prove that $x - y_0 \perp W$. Observe that for any unit vector $u \in W$ and $t \in K$ we have

$$\|x - y_0\|^2 \leq \|x - y_0 - ty\|^2 = \|x - y_0\|^2 - \bar{t}\,\langle x - y_0, y\rangle - t\overline{\langle x - y_0, y\rangle} + |t|^2.$$

As we observed in Cauchy-Schwarz inequality, take $t = \langle x - y_0, y\rangle$.

92.

Theorem 5 (Orthogonal Decomposition)**.** *Let H be a Hilbert space and $W \leq H$ be a closed vector subspace. Then $H = W \oplus W^{\perp}$.*

If $x \in H$ and $y \in W$ is such that $d(x, y) = d(x, W)$, let $z := x - y$. Then $z \perp W$ and $x = y + z$. This decomposition is unique. For, if $x = y_1 + z_1 = y_2 + z_2$ in an obvious notation, $y_1 - y_2 = z_2 - z_1$. Observe that LHS is in W and the RHS is in $W^{\perp}$. Complete the argument.

93. Refer to Item 85. What is its relevance to the last item?

94. Refer to Item 85. Let D be a dense subset of an inner product space V. If $v \perp x$ for all $x \in D$, then $v = 0$. For, let (x_n) in D converge to v. Then, by CS≤, $|\langle x_n - v, v\rangle| \to 0$..

95. Observe that $C[a, b]$ is dense in $L^2[a, b]$. Hence if $g \in L^2[a, b]$ is such that $\int_a^b fg = 0$ for all $f \in C[a, b]$, then $g = 0$ a.e. .

96. In the last item, can we replace $C[a, b]$ by P, the set of polynomials?

A side remark: Let $\mathcal{T}_1$ and $\mathcal{T}_2$ be two topologies on a set X such that $\mathcal{T}_1 \leq \mathcal{T}_2$. If D is a dense subset of $(X, \mathcal{T}_2)$, then is is also dense in $(X, \mathcal{T}_1)$.

By Weierstrass, P is dense in $C[a, b]$ under $\|\ \|_\infty$. In view of last item, it is enough to show that P is dense in $C[a, b]$ with L^2-norm. Given $f \in C[a, b]$ and $\varepsilon > 0$, let $g \in P$ be such that $\|f - g\|_\infty < \varepsilon$. Then use CS≤, to conclude that $\|f - g\|_2 \leq \varepsilon(b - a)^{1/2}$.

97. A subset A of an inner product space V is said be *orthogonal* if $\langle x, y\rangle = 0$ for $x, y \in A$ with $x \neq y$.

Note that any orthogonal set of **nonzero** vectors is linearly independent.

A subset A of an inner product space V is said be *orthonormal* if $\langle x, y\rangle = \delta_{xy}$ for $x, y \in A$. (Here $\delta_{xy} = 1$ if $x = y$ and 0 otherwise.)

98. Let $\{e_j : j \in I\}$ be an orthonormal set. Expand $\left\| x - \sum_{j \in F} \langle x, e_j\rangle\, e_j \right\|^2$ to conclude that

$$\sum_{j \in F} |\langle x, e_j\rangle|^2 \leq \|x\|^2.$$

Hence, we obtain the Bessel's inequality. .

$$\sup\left\{\sum_{j \in F} |\langle x, e_j\rangle|^2 : F \text{ a finite subset of } I\right\} \leq \|x\|^2. \tag{16}$$

Items 91–98 were done on 12-08-2011.

99. **Ex.** Let V, W be two closed subspaces of a Hilbert space H. Assume that $W \subset V^{\perp}$. Prove that the subspace $V \oplus W$ is closed. (Compare this with the subspaces in Item 80.)

100. Prove the Riesz lemma in a Hilbert space with a stronger conclusion: Let H be a Hilbert space and $V \leq H$ a closed proper vector subspace. Then there exists $u \in H$ such that $\| u \| = 1$ and $d(u, V) = 1$.

101. If $\{e_j : j \in I\}$ is an ON set in an inner product space, the set $I_x := \{j \in I : \langle x, e_j \rangle \neq 0\}$ is countable.

If I_x is uncountable, then I_x is the union of the sets $I_{x,k} := \{j \in J : |\langle x, e_j \rangle| \geq 1/k\}$. There exists $k \in \mathbb{N}$ such that $I_{x,k}$ is uncountable. Hence, $\sup \sum_F |\langle x, e_j \rangle|^2 = \infty$ where the sup is taken over all finite subsets of $I_{x,k}$. But this sum cannot exceed $\| x \|^2$, by Bessel.

Note that this proof is similar to that of the following result. If $f \in L^1(X, \mathcal{B}, \mu)$, then the set $\{x \in X : |f(x)| > 0\}$ is σ-finite.

102. We say an orthonormal set is *complete* if it is maximal among all orthonormal sets.

103. The next result characterizes complete ON sets. Any set satisfying any of the equivalent conditions below is called a (complete) *orthonormal basis.*

Theorem 6. *Let $\{e_j : j \in I\}$ be an orthonormal set in a Hilbert space H. The following are equivalent.*
(i) *$\{e_j : j \in I\}$ is a complete ON set.*
(ii) *If $x \in H$ is such that $x \perp e_j$ for $j \in I$, then $x = 0$.*
(iii) *For $x \in H$, the set $I_x := \{j \in I : \langle x, e_j \rangle \neq 0\}$ is a countable set. The series $\sum_{j \in I_x} \langle x, e_j \rangle e_j$ converges to x in H.*
(iv) *For $x, y \in H$, the set $I_{x,y} := \{j \in I : \langle x, e_j \rangle \neq 0$ or $\langle x, e_j \rangle \neq 0\}$ is countable. We have $\langle x, y \rangle = \sum_{j \in I} \langle x, e_j \rangle \cdot \overline{\langle y, e_j \rangle}$.*
(v) *For any $x \in H$, we have the Parseval identity.*

$$\| x \|^2 = \sum_j |\langle x, e_j \rangle|^2.$$

104. Note that the last theorem shows how Hilbert spaces arise. Let $\{e_j : j \in I\}$ be an orthonormal basis of a Hilbert space H. Consider the set I with the σ-algebra $\mathcal{B}$ of all subsets of I with counting measure μ. Given $x \in H$, let $f_x \colon I \to \mathbb{K}$ be given by $f_x(j) := \langle x, e_j \rangle$. The map from H to $L^2(I, \mathcal{B}, \mu)$ is an isometric linear isomorphism of H onto $L^2(I, \mathcal{B}, \mu)$.

In particular, if H has an infinite countable basis $\{e_n : n \in \mathbb{N}\}$, it is "nothing other than" ℓ^2.

105. The set $\{e_n(z) := \sqrt{n+1} z^n\}$ is an O.N. set in $L^2_H(\mathbb{D})$. We show that it is complete by showing that if $f \in L^2_H(\mathbb{D}, dA)$ is such that $\int_{\mathbb{D}} f(z) \overline{z}^n dA = 0$, then $f = 0$. up to a factor of $\frac{1}{\sqrt{\pi}}$

If f is holomorphic in $\mathbb{D}$, then we have a power series expansion of f in $\mathbb{D}$ (Why?):

$$f(z) = \sum_{m \in \mathbb{Z}_+} c_n z^m, z \in \mathbb{D}.$$

The power series is uniformly convergent on compact subsets of $\mathbb{D}$, in particular, uniformly convergent on $B(0,r) = r\mathbb{D}$. Hence

$$\int_{r\mathbb{D}} f(z)\overline{z}^n \, dA = \sum_m c_m \int_{r\mathbb{D}} z^m \overline{z}^n = c_m \pi \frac{r^{2(n+1)}}{n+1}.$$

Note that $f(z)\overline{z}^n \in L^2(\mathbb{D}) \subset L^1(\mathbb{D})$. We use DCT to take the limit $r \to 1$ and arrive at $\int_{\mathbb{D}} f(z)\overline{z}^n \, dA = \pi \frac{c_n}{n+1}$. The assumption $f \perp z^n$ now leads to us to conclude that $c_n = 0$. Hence $f = 0$.

106. **Fourier Series and a Complete ON Basis for $L^2[-\pi,\pi]$.** A good reference, which we follow closely, is Rudin's *Real and Complex Analysis*, especially §4.23–§4.26.

(a) A *trigonometric polynomial* $p(x)$ is an expression of the form $p(x) = \sum_{|k|\le n} c_k e^{ikx}$. It is said to be of *degree* n if at least one of $|c_{-n}|$ and $|c_n|$ is non-zero where $c_k \in \mathbb{C}$. Note that p is a continuous of *period* 2π (in the sense that $p(x+2\pi) = p(x)$ for all $x \in \mathbb{R}$). c_k is given by

$$c_k = \frac{1}{2\pi} \int_{-\pi}^{\pi} p(x) e^{-ikx} dx,$$

since

$$\frac{1}{2\pi} \int_{-\pi}^{\pi} e^{-irx} dx = \begin{cases} 0 & \text{if } r \neq 0 \\ 1 & \text{if } r = 0. \end{cases}$$

A trigonometric series is of the form $\sum_{-\infty}^{\infty} c_k e^{ikx}$ (just a formal expression; no assumption is made on the convergence of the series).
If $f \in L^1[-\pi,\pi]$, then the *Fourier series* of f is the trigonometric series

$$\sum_{-\infty}^{\infty} c_k e^{ikx} \text{ where } c_k := \widehat{f}(k) = \frac{1}{2\pi} \int_{-\pi}^{\pi} f(x) e^{-ikx} dx.$$

We then write $f \sim \sum \widehat{f}(k) e^{ikx}$. $\widehat{f}(k)$ are called the *Fourier coefficients* of f. Note that $|\widehat{f}(k)| \le \|f\|_{L^1[-\pi,\pi]} = \|f\|_{L^1}$.
Let $s_n(f,x) := \sum_{|k|\le n} \widehat{f}(k) e^{ikx}$ be the n-th symmetric partial sum of the Fourier series of $f \in L^1[-\pi,\pi]$.

(b) Prove the **Riemann Lebesgue Lemma:** For $f \in L^1[-\pi,\pi]$, $\lim_n \widehat{f}(n) = 0$. *Hint:* Prove this for a characteristic function of an interval $[a,b] \subseteq [-\pi,\pi]$. Use the fact that step functions are dense in $L^1[-\pi,\pi]$.

(c) A sequence $\{K_n\}$ of real valued continuous functions in $[-\pi,\pi]$ (with period 2π) is called an *approximate identity* on $[-\pi,\pi]$ if it has the following three properties:
(i) K_n is periodic and $K_n \ge 0$.
(ii) $\frac{1}{2\pi} \int_{-\pi}^{\pi} K_n = 1$.
(iii) Given $\varepsilon > 0$ and $\delta > 0$, there exists N such that if $n \ge N$ then

$$\left(\int_{-\pi}^{-\delta} + \int_{\delta}^{\pi} \right) K_n < \varepsilon$$

Geometrically, (iii) says that the area under the graph of K_n accumulates around the point 0 as $n \to \infty$.

(d) Let $\{K_n\}$ be an approximate identity on $[-\pi, \pi]$. Let f be a continuous function as $[-\pi, \pi]$ of period 2π. Then $f_n(x) := f * K_n(x) = \frac{1}{2\pi}\int_{-\pi}^{\pi} f(t)K_n(x-t)dt$ converges uniformly to f on $[-\pi, \pi]$.

$$\begin{aligned}
|f_n(x) - f(x)| &= \frac{1}{2\pi}|\int f(t)K_n(x-t)dt - \int f(x)K(t)dt| \\
&\leq \frac{1}{2\pi}|\int [f(x+t) - f(x)]K(t)dt| \\
&\leq \frac{1}{2\pi}\int |[f(x+t) - f(x)]|K(t)dt \\
&\leq \frac{1}{2\pi}\left(\int_{-\pi}^{-\delta} + \int_{\delta}^{\pi} + \int_{-\delta}^{\delta}\right),
\end{aligned}$$

where δ is chosen by uniform continuity of f. The first two terms are estimated using the bound for f and property (iii) of an approximate identity. The third is estimated using the uniform continuity of f.

(e) We now give an explicit approximate identity. Let $K_n(t) := C_n\left(\frac{1+\cos t)}{2}\right)^n$ where C_n is chosen so that $\frac{C_n}{2\pi}\int_{-\pi}^{\pi} K_nt)\,dt = 1$.

(f) Observe that K_n is even, decreasing on $[0, \pi]$ and that it satisfies the first two properties of an approximate identity (as in Item **??**).

(g) We need an upper bound for C_n, that is, a lower bound for $\int_{-\pi}^{\pi} K_n(t)\,dt$. Enough to consider the integral over $[0, \pi]$.

$$\int_0^{\pi} K_n(t)\,dt \geq \int_0^{\pi} K_n(t)\sin t\,dt \tag{17}$$

$$= \int_0^1 u^n du = \frac{1}{n+1}. \tag{18}$$

(h) To verify that the third condition in Item **??** also holds, we establish a stronger property: Given $0 < \delta < \pi$,

$$M_n(\delta) := \sup_{t\geq\delta} K_n(t) \to 0.$$

Since K_n is decreasing we have, for $t \geq \delta$,

$$K_n(t) \leq K_n(\delta) \equiv C_n\left(\frac{1+\cos\delta}{2}\right)^n \geq (n+1)r^n,$$

where $r := \frac{1+\cos\delta}{2}$. As $(n+1)r^n \to 0$ as $n \to \infty$ (why?), the result follows.

(i) Note that $p_n(t) := \frac{1}{2\pi}\int_{-\pi}^{\pi} f(s)K(s-t)\,dt$ is a trigonometric polynomial. So we have shown that the vector subspace/algebra of trigonometric polynomials is dense in $(C(\mathbb{T}), \|\ \|_\infty)$.

(j) Let $f \in \mathcal{C}([-\pi, \pi])$ be periodic. Assume $\widehat{f}(k) = 0$ for all $k \in \mathbb{Z}$. Then $f = 0$. *Hint:* Assume f to be real. Then $\int fp = 0$ for all trigonometric polynomials p. Use the last exercise to conclude that $\int f^2 = 0$.

(k) Show that the set of continuous functions in $\mathcal{C}[-\pi,\pi]$ which are periodic, i.e., $f(+\pi) = f(-\pi)$ is dense in $L^2[-\pi,\pi]$. *Hint:* Recall that $\mathcal{C}[-\pi,\pi]$ is dense in $L^2[-\pi,\pi]$. Given $g \in \mathcal{C}[-\pi,\pi] \subseteq L^2[-\pi,\pi]$, consider

$$g_n = \begin{cases} g(t) & \text{if } -\pi \le t \le t_n := \pi - \frac{1}{n^2} \\ g(t_n) - [g(-\pi) - g(t_n)] \left(\frac{t-t_n}{\pi - t_n}\right) & \text{if } t \in [t_n, \pi]. \end{cases}$$

(l) Show that the set of periodic continuous functions $C(\mathbb{T})$ is not dense in $(C[-\pi,\pi], \| \ \|_\infty)$.

(m) Is the set of trigonometric polynomials dense in $L^2[-\pi,\pi]$? If so, the last item is 'obvious'.

(n) Let $f \in L^2[-\pi,\pi]$ be such that $\widehat{f}(n) = 0$ for all $n \in \mathbb{N}$. Then $f = 0$ a.e. *Hint:* Assume f to be real. The hypothesis implies that $\int fg = 0$ for any periodic continuous function g. Use Item 106k. Or use Item 106m.

(o) For a lot of examples of approximate identities and their uses in analysis, refer to my article on "Approximate Identities".

107. V be a vector space over *some* field $\mathbb{K}$. Let $f\colon V \to \mathbb{K}$ be a non-zero linear map. Let v be such that $f(v) = 1$. Then $V = \ker f \oplus \mathbb{K}v$.

108. Recall the equivalent conditions for a linear map $T\colon X \to Y$ between two normed linear spaces to be continuous. One says a linear map $t\colon X \to Y$ is *bounded* if there exists $C > 0$ such that $\|Tx\| \le C\|x\|$ for all $x \in X$. Note that this just means that T is bounded on $B_X(0,1)$.

Note that if T is non-zero, there does **not** exist $C > 0$ such that $\|Tx\| \le C$ for all $x \in X$. Let $BL(X,Y)$ denote the set of all continuous linear maps from X to Y.

109. **Examples of continuous/bounded linear maps.**

(a) If $\dim X < \infty$, then any linear map from X to a normed linear space Y is continuous. Hence $L(X,Y) = BL(X,Y)$.

(b) The map $\Lambda\colon (\mathbf{c}, \| \ \|_\infty) \to \mathbb{K}$ given by $\Lambda(x) := \lim x_n$ is continuous.

(c) Let $(X, \mathcal{B}, \mu)$ be a measure space and $1 \le p \le \infty$. When is $T\colon L^p(X,\mathcal{B},\mu) \to \mathbb{K}$ defined by $Tf := \int_X f\,d\mu$ is a continuous linear map? If $\mu(X) < \infty$.

(d) Let $T\colon (C[0,1], \| \ \|_\infty) \to (C^1[0,1], \| \ \|_{C^1})$ given by $Tf(x) := \int_0^x f(t)\,dt$ is continuous. We have $\|Tf\|_{C^1} \le 2\|f\|_\infty$.

(e) The derivative map $D\colon (C^1[0,1], \| \ \|_{C^1}) \to (C[0,1], \| \ \|_\infty)$ given by $Df := f'$ is continuous.

But, $D\colon (C^1[0,1], \| \ \|_\infty) \to (C[0,1], \| \ \|_\infty)$ given by $Df := f'$ is **not** continuous. (Note the norm on the domain.)

(f) **Integral Operators.** Let $I = [a,b]$ be a closed and bounded interval of $\mathbb{R}$. Let $k(x,y) \in C(I \times I)$. Define $T_k\colon C(I) \to C(I)$ by $Tf(x) := \int_a^b k(x,y)f(y)\,dy$. Then T_k is a bounded/continuous linear operator on $(C(I), \| \ \|_\infty)$. The function k is called the *kernel* of the *integral* operator T_k. (Do not confuse the kernel k with the kernel of the linear map T_k!)

(g) A linear map $T\colon X \to \mathbb{K}$ is called a linear *functional*. Consider $\Lambda\colon \ell^1 \to \mathbb{K}$ defined by $\Lambda(z) := \sum_n z_n$. Then Λ is a continuous linear functional on ℓ^1.

(h) Fix $a \in \ell^1$. Consider $T\colon \mathbf{c_0} \to K$ defined by $T(z) := \sum_n a_n z_n$. Then T is a continuous linear functional.

(i) The map $T\colon \mathbf{c}_{00} \to \mathbb{K}$ defined by $Tz = \sum_n z_n$ is linear but not continuous.

(j) Let P denote the set of polynomials considered as a vector subspace of $(C[0,1], \|\ \|_\infty$. Let $\{p_i\}$ be a basis of P and let it be extended to a basis $\mathcal{H} := \{p_i\} \cup \{f_0\} \cup \{f_j\}$ of $c[0,1]$. Define a new norm $\|\ \|_{\mathcal{H}}$ on $C[0,1]$ as follows:

$$\| f \|_{\mathcal{H}} := \sum_i |a_i| + |b_0| + \sum_j |b_j| \text{ where } f = \sum_i a_i p_i + b_0 f_0 + \sum_j b_j f_j.$$

Note that the sums above are finite so that $\| f \|_{\mathcal{H}}$ is well-defined.
Define a linear map $T\colon C[0,1] \to \mathbb{K}$ by setting $Tp_i = 0$ for all i, $Tf_0 = 1$ and $Tf_j = 0$ for all j and extend this linearly to all of $C[0,1]$. Then $T\colon (C[0,1], \|\ \|_{\mathcal{H}}) \to \mathbb{K}$ is continuous whereas $T\colon (C[0,1], \|\ \|_\infty) \to \mathbb{K}$ is **not** continuous.

(k) Let X be a compact space. Consider the evaluation map $Tf := f(t_0)$ where $f \in (C(X), \|\ \|_\infty)$ and $t_0 \in X$. Then T is a continuous linear functional.
Can such a 'map' be defined on L^p spaces?

(l) **Multiplication Operator.** Let $(X, \mathcal{B}, \mu)$ be a measure space. Let $\varphi \in L^\infty$. Let $1 \le p \le \infty$. Consider $M_\varphi\colon L^p \to L^p$ defined by $M_\varphi f = f\varphi$. Then M_φ is a continuous linear map from L^p to L^p.

(m) We recalled the Riesz representation theorem in a finite dimensional inner product space. We gave two proofs of this. The second proof can be adapted to prove the following.

Theorem 7 (Riesz Representation Theorem)**.** *Let H be a Hilbert space. Then any continuous linear functional T on H is of the form $Tx = \langle x, v\rangle$ for a unique vector $v \in H$.*

Assume $T \neq 0$. Let $W := \ker T$. Then $H = W \oplus W^\perp$ is the orthogonal decomposition. Note that $W^\perp$ is one-dimensional. Choose $y \in W^\perp$ such that $Ty = 1$. Then the required v must be of the form $v = cy$ for some $c \in \mathbb{K}$. (Why?) How to identify c?

(n) Let H be a Hilbert space and W be a closed vector subspace of H. Let $H = W \oplus W^\perp$ be the orthogonal decomposition of H. Let $P_W\colon H \to W \subset H$ be the orthogonal projection defined by $P_W(x) = w$ where $x = w + w^\perp \in W \oplus W^\perp$. Then P_w is a continuous linear map.

(o) We combine the ideas of Item 109l and Item 109n. Fix $\varphi \in L^\infty(\mathbb{D})$. Consider the map $T\colon f \mapsto M_\varphi(f) \mapsto P_{L^2_H(\mathbb{D})}(M_\varphi(f))$. Then T is a continuous linear map.

(p) Let $g \in L^q(X, \mathcal{B}, \mu)$. Consider $T\colon f \mapsto \int_X fg$ for $f \in L^p$. Then T is a continuous linear functional on L^p.

(q) Let H be a Hilbert space with an O.N. basis $\{e_n : n \in \mathbb{N}\}$. Write $x = \sum_n x_n e_n$. (Do you understand this?) Let (a_n) be a sequence in $\mathbb{K}$. Define $Tx := \sum_n a_n x_n$. When does $\sum_n a_n x_n \in H$? A sufficient condition is $a \in \ell^\infty$. In such a case, T is a continuous linear map.

(r) Define $T\colon \mathbf{c} \to \mathbf{c}$ by setting $Tx = a$ where $a_n := (x_1 + \cdots + x_n)/n$. (Why does $a \in \mathbf{c}$?) Is T linear?

(s) **Shift Operators.** The right shift operator $R\colon \ell^2 \to \ell^2$ is defined as $R((z_n)) = (0, z_1, z_2, \ldots)$. Then R is a linear isometry but not onto.
The left shift L is defined by $L((z_n)) = (z_2, z_3, \ldots)$. Then L is bounded, linear, onto but not injective. Note also that $L \circ R$ is the identity while $R \circ L$ is not.

110. **Operator Norm.** Let X, Y be normed linear spaces. Let $T\colon X \to Y$ be a bounded linear map. This means that there exists $C \geq 0$ such that for $x \in X$, we have $\|Tx\| \leq C\|x\|$. We let $\|T\|$ denote

$$\|T\| := \inf\{C : \forall x \in X, \|Tx\| \leq C\|x\|\}.$$

We call $\|T\|$ as the operator norm of T. It is easy to show that that $T \mapsto \|T\|$ is a norm on $BL(X,Y)$. Note that to find the norm of T we need estimate $\|Tx\|$ in a very *efficient* way.

111. Exercise. Operator norm can be defined in equivalent ways:

$$\begin{aligned}
\|T\| &= \inf\{C : \forall x \in X, \|Tx\| \leq C\|x\|\}.\\
&= \inf\{C : \forall x \in B(0,1), \|Tx\| \leq C\}.\\
&= \inf\{C : \forall x \in B[0,1], \|Tx\| \leq C\}.\\
&= \inf\{C : \forall x \in X \text{ with } \|x\| = 1, \|Tx\| \leq C\}.\\
&= \sup\{\|Tx\| : \|x\| \leq 1\}\\
&= \sup\{\|Tx\| : \|x\| = 1\}\\
&= \sup\left\{\frac{\|Tx\|}{\|x\|} : \|x\| \neq 0\right\}.
\end{aligned}$$

112. **Examples of operator norms.**

(a) If T is the identity operator on X, then $\|T\| = 1$. More generally, $\|\lambda I\| = |\lambda|$.

(b) Let $T\colon \mathbb{C}^2 \to \mathbb{C}^2$ be the diagonal operator defined by $T(z_1, z_2) = (az_1, bz_2)$. $\mathbb{C}^2$ is equipped with the standard Hermitian inner product. Then $\|T\| = \max\{|a|, |b|\}$.

(c) Let A be an Hermitian $n \times n$-matrix. $\mathbb{C}^n$ is equipped with the standard Hermitian inner product. and considered as column vectors or $n \times 1$-matrices. Let $T\colon \mathbb{C}^n \to \mathbb{C}^n$ be defined by $Tz = Az$ where Az is the matrix multiplication. Then

$$\|T\| = \max\{|\lambda| : \lambda \text{ is an eigenvalue of } A\}.$$

Recall that A is a diagonalizable with real eigenvalues. Let $\{u_j : 1 \leq j \leq n\}$ be an ON basis of $\mathbb{C}^n$ such that $Tu_j = \lambda_j u_j$, $1 \leq j \leq n$. Write $z = \sum_j \xi_j u_j$. What is Tz in terms of this expression? Use Pythagoras theorem.

(d) The shift operators have norm 1.

(e) Let X be a compact Hausdorff space. Let $\varphi \in C(X)$. The *multiplication operator* is defined as $M_\varphi(f) = f\varphi$. Then $\|M_\varphi\| = \|\varphi\|_\infty$.

(f) Let $(X, \mathcal{B}, \mu)$ be a σ-finite measure space. Let $\varphi \in L^\infty(X, \mathcal{B}, \mu)$. We define a multiplication operator M_φ as earlier but now on $L^2(X, \mathcal{B}, \mu)$. It is easy to verify that M_φ is a bounded linear operator on L^2 with $\|M_\varphi\| \leq \|\varphi\|_\infty$. We show that equality holds in this.

Assume first that $\mu(X) < \infty$. If $\alpha < \|\varphi\|_\infty$, by the very definition of essential supremum, it follows that $E := \{x \in X : |\varphi(x)| \geq \alpha\}$ has positive measure. Note that $\mu(E) < \infty$. If $f := \frac{1}{\sqrt{\mu(E)}}\chi_E$, then $\|f\|_2 = 1$ and we have $\|M_\varphi(f)\|_2^2 \geq \alpha^2$. Hence we conclude that $\|M_\varphi\| \geq \alpha$. Extend this argument to the σ-finite case.

(g) Let H be a Hilbert space. Fix $v \in H$. Then $f_v(x) := \langle x, v\rangle$ is a continuous linear functional and we have $\|f_v\| = \|v\|$.

(h) Let A be an $n \times n$-matrix with entries in $\mathbb{K}$. Consider the linear map T defined by A on $\mathbb{K}$ (as in Item 112c). Let the norm on $\mathbb{K}^n$ be the max norm $\|\ \|_\infty$. Then $\|T\| = \max_{1\leq i\leq n}\{\sum_{j=1}^n |a_{ij}|\}$.

(i) Let A be an $n \times n$-matrix with entries in $\mathbb{K}$. Consider the linear map T defined by A on $\mathbb{K}$ (as in Item 112c). Now, let the norm on $\mathbb{K}^n$ be $\|\ \|_1$. Then $\|T\| = \max_{1\leq j\leq n}\{\sum_{i=1}^n |a_{ij}|\}$.
Note the way the indices run in the last two expressions for $\|T\|$.

113. Exercises.

(a) Let X be a normed linear space and $f\colon X \to \mathbb{K}$ be linear. If f is not continuous, then $f(B_X(0,1)) = \mathbb{K}$.

(b) Let the notation be as in the last item. Show that f is continuous iff the kernel of f is closed.

To prove the non-trivial part, assume that $f \neq 0$. Let us choose $a \notin \ker f$ such that $f(a) = 1$. There exists $r > 0$ such that $B(a,r) \cap \ker f = \emptyset$. Let $x \notin \ker f$. Then $a - \frac{x}{f(x)} \in \ker f$ and hence $a = \frac{x}{f(x)} \notin B(a,r)$. That is,

$$\left\|\left(a - \frac{x}{f(x)}\right) - a\right\| \geq r \implies \frac{\|x\|}{|f(x)|} \geq r.$$

Consequently, we obtain $|f(x)| \leq \frac{1}{r}\|x\|$ for $x \notin \ker f$. Since this inequality is true even for $x \in \ker f$, we are through.

(c) A linear functional $f\colon X \to \mathbb{K}$ is not continuous iff $\ker f$ is a proper dense subset of X.

Let $\ker f$ be a proper dense subspace. If f were continuous, then $\ker f$ is closed. Hence we are forced to conclude that $\ker f = X$. (Why?)

Let f be non-continuous. Then $\ker f$ cannot be X. (Why?) and hence it is a proper subspace of X. If $\ker f$ is not dense, then there exists $B(a,r) \cap \ker f = \emptyset$. Note that $a \notin \ker f$. We now argue as in the last item to conclude that f is continuous.

114. Let X be any normed linear space and Y be Banach. Then $BL(X,Y)$ is a Banach space under the operator norm.

Sketch: Let (T_n) be Cauchy. For any $x \in X$, (T_nx) is Cauchy in Y and hence converges to an element in Y. Denote the limit by Tx. Then $x \mapsto Tx$ is linear continuous and $\|T_n - T\| \to 0$. T is additive:

$$T(x+y) = \lim T_n(x+y) = \lim(T_nx + T_ny) = \lim T_nx + \lim T_ny = Tx + Ty.$$

T is bounded: Let M be such that for $n \in \mathbb{N}$, we have $\|T_n\| \leq M$. For any given $\varepsilon > 0$, let $N_x \in \mathbb{N}$ be such that for $n \geq N_x$, $\|Tx - T_nx\| \leq \varepsilon$. For $x \in B[0,1]$, we have for $n \geq N_x$

$$\|Tx\| \leq \|Tx - T_nx\| + \|T_nx\| \leq \varepsilon + M.$$

$\|T_n - T\| \to 0$: Again curry-leaves trick. For given $\varepsilon > 0$, choose n_0 to exploit the Cauchy nature of (T_n). Let N_x be as above. For $x \in B[0,1]$, we have

$$\|T_nx - Tx\| \leq \|T_nx - T_mx\| + \|T_mx - Tx\| \leq \varepsilon/2 + \varepsilon/2,$$

where $n \geq n_0$ and the curry-leaf m is chosen so that $m \geq \max\{n_0, N_x\}$.

115. An important corollary of the last item is: For any normed linear space X, the dual $X^* = BL(X, \mathbb{K})$ is complete.

116. Let X and Y be normed linear spaces. Assume Y is complete. Let $D \subset X$ be dense vector subspace. Assume that $T\colon D \to Y$ be a continuous linear map. Then there exists a unique continuous linear map $\tilde{T}\colon X \to Y$ such that $\tilde{T}(x) = Tx$ for $x \in D$.

    $\tilde{T}$ is called the continuous extension of T. In future, we shall denote $\tilde{T}$ by T. It is easy to see that $\left\|\tilde{T}\right\| = \|T\|$. (Understand how the norms in this equality are defined.)

    Note that this is a special case of the standard result from metric spaces. While the proof is easy and straightforward, it is often used in analysis.

117. Let $A\colon X \to Y$ and $B\colon Y \to Z$ be continuous linear maps between normed linear spaces. Then it is easy to see that $\|B \circ A\| \leq \|A\| \|B\|$.

    Strict inequality can occur. (Recall nilpotent linear maps exist!)

118. An important special case of the last item: If $T \in BL(X)$, then $\|T^n\| \leq \|T\|^n$.

119. The last item and geometric series suggests the following question. Let X be a normed linear space and $A \in BL(X)$. Does the series $\sum_n A^n$ converge if $\|A\| < 1$, and if so, what is the sum?

    Recall that in a Banach space any norm convergent series is convergent. So, if we assume X to be complete so that $BL(X)$ is complete, we know that the series $\sum_{n=0}^{\infty} A^n$ is convergent. It is no surprise that the sum should be the inverse of $(I - A)$. For,

$$(I - A)(I + A + \cdots + A^n) = I - A^{n+1} \to I \text{ in } BL(X).$$

120. The crucial fact to observe in the last item is that if $\|A\| < 1$, $(I - A)$ is invertible as a linear map *and* that the inverse is continuous.

121. We used the result of Items 119–120 to solve a Volterra type integral equation.

    Let $K \in C([0,1] \times [0,1])$ and $g \in C[0,1]$. We wish to solve the Volterra integral equation

$$f(x) - \int_0^x K(x,y) f(y)\, dy = g(x), \quad \text{for } x \in [0,1].$$

Observe that the equation can be expressed as $(I-T)f = g$, where $Tf(x) := \int_0^x K(x,y)f(y)\,dy$. It is easy to see that $T\colon C[0,1] \to C[0,1]$ is continuous linear.

$$|Tf(x)| \leq \int_0^x |K(x,y)||f(y)|\,dy \leq x \, \|K\|_\infty \, \|f\|_\infty \,.$$

(Note that we do not use to this to get an estimate for $\|T\|$!) By induction, it follows that

$$|T^n f(x)| \leq \frac{x^n}{n!} \, \|K\|_\infty \, \|f\|_\infty \,.$$

Hence we conclude that $\|T^n\| \leq \frac{\|K\|_\infty}{n!}$. Thus, the series $\sum_{n=0}^\infty T^n$ converges so that $(I-T)^{-1}$ exists as a bounded linear operator. We therefore conclude the integral equation has a unique solution given by $f = \sum_{n=0}^\infty K^n g$. Note that we also have an estimate for the solution in terms of the given data:

$$\|f\|_\infty = \left\| \sum_{n=0}^\infty K^n g \right\| \leq \sum_{n=0}^\infty \|K^n\| \, \|g\|_\infty \leq \sum_{n=0}^\infty \left(\frac{\|K\|_\infty}{n!} \right) \|g\|_\infty = e \, \|K\|_\infty \, \|g\|_\infty \,.$$

122. Exercise. Let X be a Banach space. Show that the set

$$GL(X) := \{T \in BL(X) : T^{-1} \text{ exists and lies in } BL(X)\}$$

is an open subset in $BL(X)$.

123. In the next few lectures, we shall deal with the four pillars of functional analysis and their applications. The pillars are are the Hahn-Banach Theorem (HBT, Thm. 11), the Uniform Boundedness Principle (UBP, Thm. 13, Item 147), Open Mapping Theorem (OMT, Thm. 18, Item 162) and the Closed Graph Theorem (CGT, Thm. 20, Item 169).

These are the results which make Functional Analysis as a veritable tool for any analyst. The reader is advised to solve as many exercises as possible to master the use of these pillars.

124. The main aim of Hahn-Banach theorem is to show that there exists an abundant supply of continuous linear functionals on any normed linear space. See Prop. 136 in Item 135. This is very easy in the case of a Hilbert space. See Item 128.

125. Note that in the context of topology, similar questions are raised. If X is a metric space, then the collection of functions $f_p\colon x \mapsto d(x,p)$ "separates points", in the sense that given $x_1 \neq x_2$, there exists a real valued continuous function f on X such that $f(x_1) \neq f(x_2)$. To make such an assertion in the context of general topology, we require the space to be Hausdorff and either completely regular or normal.

126. Hahn-Banach Theorems says that if f is a continuous linear functional on a vector subspace Y of a normed linear space X, then there is a continuous linear functional $\tilde{g}$ on X such that (i) $\tilde{g}(y) = g(y)$ for $y \in Y$ and (ii) $\|g\| = \|\tilde{g}\|$.

There are two easy cases. If Y is dense in X, then Item 116 yields the result. If X is a Hilbert space, then we can extend g uniquely to $\overline{Y}$ by Item 116. Let $X = \overline{Y} + Y^\perp$ be the orthogonal direct sum. Set $\tilde{g}(z) = 0$ if $z \in Y^\perp$.

127. Let V be a vector space over a filed k. Let $f: V \to k$ be a nonzero linear map. Let $W := \ker f$. Show that W is of codimension 1 in the sense that if w' is a vector subspace such that $V = W \oplus W'$, then $\dim W' = 1$.

128. **(Riesz Representation Theorem)**

Theorem 8. *Let H be a Hilbert space. Let $f: H \to \mathbb{K}$ be a continuous linear functional. Then there exists a unique $u \in H$ such that $f(x) = \langle x, u \rangle$ for all $x \in H$.*

Proof. Let $f \neq 0$. Consider the null space Y of f. Choose $z \in Y^\perp$ with $f(z) = 1$. Note that $\dim W^\perp = 1$. Choose c suitably so that $u = cz$ is as required. Uniqueness is trivial.

Assuming one such u exists, let us write it in terms of an orthonormal basis $\{e_i\}$: $u := \sum u_i e_i$. Then

$$f(e_i) = \langle e_i, u \rangle = \left\langle e_i, \sum_j u_j e_j \right\rangle = \overline{u_i}.$$

□

129. A sublinear or a Minkowski functional on a real vector space X is a function $p: X \to \mathbb{R}$ satisfying

(a) $p(x + y) \leq p(x) + p(y)$ for $x, y \in X$.
(b) $p(\alpha x) = \alpha p(x)$, for $x \in X$ and $\alpha \geq 0$.

130. We start with a lemma preliminary to Hahn-Banach.

Lemma 9. *Let Y be a vector subspace of a real vector space X and p a sublinear functional on X. Let $f: Y \to \mathbb{R}$ be a linear map such that $f(y) \leq p(y)$ for $y \in Y$. Then f has an extension to a linear functional $F: X \to \mathbb{R}$ such that $F(x) \leq p(x)$ for $x \in X$.*

Proof. Observe that if g is a linear functional extending f to a subspace Z containing Y such that $g(z) \leq p(z)$ for $z \in Z$, we must have for $z_0 \in Z \setminus Y$

$$\alpha := \sup_{z_1 \in Z} \{g(z_1) - p(z_1 - z_0)\} \leq \inf_{z_2 \in Z} \{p(z_2 + z_0) - g(z_2)\} =: \beta.$$

Let λ be between α and β. Then if we define $g(y + rz_0) = f(y) + r\lambda$, then g extends f to a linear functional on span $[Y \cup \{z_0\}]$ satisfying $g(z) \leq p(z)$.

We now apply Zorn's lemma to complete the proof. □

131.

Theorem 10 (Hahn-Banach Theorem—Real Version)**.** *Let X be a normed linear space over $\mathbb{R}$ and Y be any linear subspace. Let $f: Y \to \mathbb{R}$ be a continuous linear functional. Then there exists $g \in X^*$ such that $g(y) = f(y)$ for $y \in Y$ and $\|g\| = \|f\|$.*

132. Let X be a complex vector space. Let f be a complex linear functional on X. Write $f(x) = u(x) + iv(x)$ in the usual way. Then u is a real linear functional on X and we have $f(x) = u(x) - iu(ix)$. For, $f(ix) = u(ix) + iv(ix)$, but we also have $f(ix) = if(x) = i(u(x) + iv(x))$.

Conversely, if u is any real linear functional and if we define $f(x) := u(x) - iu(ix)$ then f is complex linear functional on X.

133. Let the notation be as above. Assume that X is a normed linear space. Show that $\| u \| = \| f \|$. *Hint:* Use the *standard trick*. Choose $e^{i\theta}$ so that

$$e^{i\theta} f(x) = |f(x)| = f(e^{i\theta} x) = u(e^{i\theta} x) \leq \| u \| \, \| x \| .$$

134.

Theorem 11 (Hahn-Banach Theorem—Complex Version)**.** *Let X be a normed linear space over $\mathbb{C}$ and Y a linear subspace of X. Let $f\colon Y \to \mathbb{C}$ be a bounded linear functional. Then there exists $g \in X^*$ such that $g(y) = f(y)$ for all $y \in Y$ and $\| g \| = \| f \|$.*

Use the last two items along with the real version.

135. The following proposition contains the most important consequences of Hahn-Banach Theorem and are often used in the sequel without any specific reference.

136. Let X be a normed linear space.

(1) If $0 \neq x \in X$, then there exists $f \in X^*$ such that $\| f \| = 1$ and $f(x) = \| x \|$. In particular, $\| x \| = \sup\{|f(x)| : \| f \| \leq 1\}$.

(2) The bounded linear functionals on X separate points of X, i.e., given two distinct points $x, y \in X$, there exists an $f \in X^*$ such that $f(x) \neq f(y)$.

(3) If Y is a closed linear subspace of X and $x \in X \setminus Y$, there exists an $f \in X^*$ such that $\| f \| = 1$, $f = 0$ on Y and $f(x) = d(x, Y)$.

Proof. To prove (1), consider $f_0\colon \mathbb{K}x \to \mathbb{K}$ given by $f_0(\lambda x) = \lambda \| x \|$. Then $\| f_0 \| = 1$. By Hahn-Banach, there exists an extension f to X which does the job. The rest of (1) and (2) are left as exercise.

To prove (3), define $f_0\colon Y + \mathbb{K}x \to \mathbb{K}$ by setting $f_0(y + \lambda x) = \lambda\delta$ where $\delta = d(x, Y)$. Now, Y is closed in $Y + \mathbb{K}x$ and $\ker f_0 = Y$ is closed and hence f_0 is continuous. Now proceed as in (1).

Alternatively, $\overline{x} \in X/Y$ is nonzero and hence there exists $\varphi\colon X/Y \to \mathbb{K}$ continuous linear functional such that $\| \varphi \| = 1$ and $\varphi(\overline{x}) = d(x, Y)$. Then $\varphi \circ \pi$ meets our requirements. □

137. Hahn-Banach theorem is a most useful tool in Analysis. It is the ingenuity of the person to create situations in which Hahn-Banach may be invoked. Do not underestimate any theorem which assures existence of something.

138. **Exercises.**

(a) Let X be a nonzero normed linear space. Assume that $BL(X, Y)$ is complete under the operator norm. Then Y is complete. *Hint:* Fix a nonzero $f \in X^*$ and x such that $f(x) = 1$. For a given Cauchy sequence y_n, let $T_n(x) := f(x)y_n$.

(b) Let X be a normed linear space. Then X is isometrically isomorphic to a closed subspace of $B(\Lambda, \mathbb{C})$ for some set Λ. *Hint:* Let $\{x_\alpha : \alpha \in \Lambda\}$ be a dense subset of B_X. Let $f_\alpha \in X^*$ be such that $f_\alpha(x_\alpha) = \| x_\alpha \|$ and $\| f_\alpha \| = 1$. Let $\varphi\colon X \to B(\Lambda)$ be defined by $\varphi(x)(\alpha) := f_\alpha(x)$.

(c) If X is a separable normed linear space, then X is isometrically isomorphic to a subspace of ℓ^∞.

(d) Let X be a normed linear space such that X^* is separable. Then X is separable. *Hint:* Let $\{f_n\}$ be dense in $S_{X^*} = \{f \in X^* : \|f\| = 1\}$. Choose $x_n \in B_X$ such that $2|f_n(x_n)| \geq 1$. Show that the closure of the linear span of $\{x_n\}$ is X.

(e) Show that there exists $g \in X^*$ such that g is not of the form $f(x) = \sum_n a_n x_n$ for some $a \in \ell^1$. *Hint:* Start with $f \colon \mathbf{c} \to \mathbb{C}$ given by $f(x) := \lim_n x_n$.

(f) Give another proof of Riesz lemma along the following lines: Choose $z \in X \setminus Y$. Define $f \colon Y + \mathbb{K}z \to \mathbb{K}$ by setting $f(y + \lambda z) = \lambda$. Extend f to $F \in X^*$. Given $\varepsilon > 0$, choose an $x \in X$ with $\|x\| = 1$ so that $\|F(x)\| \geq (1 - \varepsilon)\|F\|$. Show that this x does the trick.

(g) Let X be an infinite dimensional Banach space. Prove that there exists an infinite strictly decreasing sequence (Y_n) of infinite dimensional closed linear subspaces of X. *Hint:* Take Y_1 to be the null space of $f_1 \in X^*$.

(h) Let X be an infinite dimensional Banach space. Prove that the vector space ℓ^∞ is isomorphic to a linear subspace of X. *Hint:* Take $x_n \in Y_{n-1} \setminus Y_n$ (in the notation of the last exercise) such that $\|x_n\| < 2^{-n}$. Consider the map $(a_n) \mapsto \sum a_n x_n$.

(i) Prove that the dimension of an infinite dimensional Banach space is greater than or equal to $c = 2^{\aleph}$, the cardinality of $\mathbb{R}$.

(j) A Banach space is finite dimensional iff every linear subspace is closed. *Hint:* Consider the linear span of x_n's of Ex. 138h.

(k) Let X be a complex Banach space and U an open set in $\mathbb{C}$. A map $x \colon U \to X$ is said to be *analytic* on U if

$$\lim_{h \to 0} \frac{\|x(z+h) - x(z)\|}{\|h\|} \text{ exists for all } z, z+h \in U.$$

If $f \in X^*$ and x is analytic on U, then $f \circ x$ is an analytic function from U to $\mathbb{C}$ in the standard sense.

(l) If $x \colon \mathbb{C} \to X$ is analytic and bounded in the sense that there exists $M > 0$ such that $\|x(z)\| \leq M$ for all $z \in \mathbb{C}$, then x is a constant. This is the vector analogue of *Liouville's theorem*

139. Let X be any vector space over a field. Let X' denote its dual, the vector space of linear functionals on X. Let X'' denote the dual of X'. We have a natural map of X into X'' given by $x \mapsto \tilde{x}$ where $\tilde{x}(f) = f(x)$ for $f \in X'$. If X is finite dimensional it is easy to see that X'' is linear isomorphic to X.

140. Let X be a normed linear space and $X^* \equiv BL(X, \mathbb{K})$ denote the normed linear space of continuous linear functionals on X. Let X^{**} denote the dual of X^*. Analogous to the natural map of X to X'' in the last item, we have a map $x \mapsto \tilde{x}$ of X to X^{**} defined the same way. Clearly, $\tilde{x} \in X^{**}$ and we have $\|\tilde{x}\| \leq \|x\|$. In view of Proposition 136, we see that this map is a (linear) isometry. Thus, we have an isometric embedding of X into its double dual X^{**}.

141. If the natural isometric embedding is onto, then the normed linear space X is said to be *reflexive*.

Two easy examples are (i) finite dimensional normed linear spaces and (ii) Hilbert spaces. Note that Riesz representation theorem for Hilbert spaces says that H and H^*

are "isometric" by conjugate linear map. If $f \in H^*$, then $f = f_v$ and hence the map $f \mapsto v$ is additive, isometric and we have $\lambda f \mapsto \overline{\lambda} v$ if $f = f_v$. (This is in the case of $\mathbb{K} = \mathbb{C}$. If $\mathbb{K} = \mathbb{R}$, then $H^* = H$.)

We shall take up more examples and some non-examples of this phenomenon later.

142. An immediate application of the isometric embedding of X into its double dual is the existence of a *completion* of X.

Let $\varphi \colon X \to X^{**}$ be the natural isometric embedding. If we let Y the closure of $\varphi(X)$ in the Banach space X^{**}, then Y is complete. Hence the map $\varphi \colon X \to Y$ is an isometry of X and the image $\varphi(X)$ is dense in the complete space Y. Hence the ordered triple (φ, X, Y) is a completion of the space X.

143. Let us make a useful observation about reflexive Banach spaces. For any normed linear space X, we have, in view of Hahn-Banach,

$$\| x \| = \sup\{|f(x)| : f \in X^*, \| f \| = 1\} = \max\{|f(x)| : f \in X^*, \| f \| = 1\}.$$

For $f \in X^*$, we have

$$\| f \| = \sup\{|f(x)| : x \in X, \| x \| = 1\}.$$

In general, here supremum cannot be replaced by maximum. *But* if X is reflexive, we obtain

$$\| f \| = \sup\{|f(x)| : x \in X, \| x \| = 1\}. = \max\{|f(x)| : x \in X, \| x \| = 1\}.$$

144. We use the last item to show that $\mathbf{c_0}$ is not reflexive. Consider $\varphi \colon \mathbf{c_0} \to \mathbb{C}$ given by $\varphi(z) := \sum_{n=1}^{\infty} \frac{z_n}{n!}$. Then φ is a continuous linear functional on $\mathbf{c_0}$ with $\| \varphi \| = e - 1$. But it is easy to show that for any $z \in \mathbf{c_0}$ with $\| z \| = 1$, we have $|\varphi(z)| < e - 1$. Hence $\mathbf{c_0}$ cannot be reflexive.

145. The next two results, namely, Uniform boundedness principle and Open mapping theorem need the following.

146. We recall Baire's theorem.

Theorem 12. *Let (X, d) be a complete metric space. Let $X = \cup_n F_n$ be a countable union of closed sets. Then at least one F_n has a non-empty interior.* □

147. Uniform Boundedness Principle (UBP) — Banach-Steinhaus

Theorem 13. *Let X be a Banach space and Y a nls. Let $\{T_\alpha : \alpha \in I\} \subset BL(X, Y)$ be given. Assume that for each $x \in X$, there exists C_x such that $\sup_I \{\| T_\alpha(x) \|\} \le C_x$. Then there exists M such that $\| T_\alpha \| \le M$ for $\alpha \in I$.*

Proof. Use Baire's theorem. Consider $F_n := \{x \in X : \| Tx \| \le n \text{ for all } T\}$. Then $X = \cup F_n$. □

We give two more proofs of UBP which avoid Baire's theorem and introduce important analytic tricks.

148. For the 2nd proof of UBP, we first make an observation.

Lemma 14. *Let X, Y be normed linear spaces. Let $T \in BL(X,Y)$. Then*

$$\|T\| r \leq \sup_{x \in B(a,r)} \|Tx\| \leq \|Ta\| + r\|T\|. \tag{19}$$

Proof. Observe for any $v = ru \in B(0,r) = rB(0,1)$, we have

$$\max\{\|T(a+v)\|, \|T(a-v)\|\} \geq \frac{1}{2}(\|T(a+v)\| + \|T(a-v)\|) \geq \|Tv\| = r\|Tu\|.$$

Taking supremum over $u \in B(0,1)$ yields the left most inequality. The other one is easy. □

149. 2nd Proof of UBP. Assume $\sup\{\|T_\alpha\|\} = \infty$. Choose T_n from this family so that $\|T_n\| \geq n4^n$. Let $x_0 = 0$. We use induction and the lemma of the last item to choose x_n such that (i) $\|x_{n-1} - x_n\| \leq 4^{-n}$ and (ii) $\|T_n x_n\| \geq \frac{1}{2}\|T_n\| 4^{-n}$. For $n \geq m$ we have

$$\|x_n - x_m\| \leq \|x_n - x_{n-1}\| + \cdots + \|x_{m+1} - x_m\| \leq 4^{-n} + \cdots + 4^{-(m+1)}.$$

It follows that (x_n) is Cauchy. Since X is complete, there exists $x \in X$ such that $x_n \to x$. We claim that $\sup_n \|T_n x\| = \infty$, contrary to our hypothesis.

The argument which showed that (x_n) is Cauchy can be adapted to show that $\|x - x_n\| \leq \frac{1}{3}4^{-n}$. Hence we obtain

$$\begin{aligned} \|T_n x\| &= \|T_n x - T_n x_n + T_n x_n\| \\ &\geq \|T_n x_n\| - \|T_n(x - x_n)\| \\ &\geq n4^{-n}\|T_n\| - \|T_n\| \frac{1}{3}4^{-n} \\ &= \frac{2}{3}4^{-n}\|T_n\| \\ &\geq \frac{2}{3}4^{-n} n4^n. \end{aligned}$$

This completes the proof. □

150. We now indicate a third proof. This proof employs a very useful technique known as Gliding Hump Method.

Idea of the proof. Assume, as earlier, $\sup\{\|T_\alpha\|\} = \infty$. We inductively choose T_n and x_n such that $\|x_n\| = 4^{-n}$ and $\|T_n x\| \geq n$ where $x = \sum_n x_n$. Look at

$$T_n x = T_n(x_1 + \cdots + x_{n-1}) + T_n x_n + T\left(\sum_{k=n+1}^{\infty} T_n x_k\right).$$

As we wish to have $\|T_n x\| \geq n$, we need a lower bound for it. The obvious idea is to exploit

$$\|T_n x\| \geq \|T_n x_n\| - \|T_n(x_1 + \cdots + x_{n-1}\| - \left\|T_n\left(\sum_{k=n+1}^{\infty} x_k\right)\right\|.$$

That is, the term $\|T_n x_n\|$ should dominate the other two terms exceedingly well. To achieve this, we should estimate upper bounds for these two terms.

Let us deal with the finite sum first. By hypothesis, there exists C_{n-1} such that $\|T_\alpha(x_1 + \cdots + x_{n-1}\| \leq C_{n-1}$ for any α, in particular, for $\alpha = n$.

Since we intend to use induction the vectors x_k for $k > n$ are not yet known to us, except their norms. Hence the only fact available to us is

$$\left\| T_n \left(\sum_{k=n+1}^{\infty} x_k \right) \right\| \leq \|T_n\| \left\| \sum_{k=n+1}^{\infty} x_k \right\| \leq \|T_n\| \sum_{k=n+1}^{\infty} \|x_k\| = \|T_n\| \frac{1}{3} 4^{-n}.$$

If we remember that $4^{-n} = \|x_n\|$, the above estimate acquires a different dimension:

$$\left\| T_n \left(\sum_{k=n+1}^{\infty} x_k \right) \right\| \leq \|T_n\| \frac{1}{3} 4^{-n}.$$

Now everything has fallen into place. We need to make sure that $\|T_n x_n\| \geq \frac{2}{3} \|T_n\| \|x_n\|$ so that after taking away $\left\|T_n \left(\sum_{k=n+1}^{\infty} x_k\right)\right\|$, we are still left with $\frac{1}{3} \|T_n\| \|x_n\|$. But from this we need to take away C_{n-1}.

We thus end up with an estimate of the following kind.

$$\|T_n x\| \geq \frac{1}{3} \|T_n\| \|x_n\| - C_{n-1}.$$

We wish to have the right side to be greater than n. That is, $\|T_n\| 4^{-n} \geq 3(n + C_{n-1})$ or $\|T_n\| \geq 3 \times 4^n (n + C_{n-1})$.

We are now ready for the textbook proof. Choose $T_1 \in \{T_\alpha\}$ such that $\|T_1\| \geq 3 \times 4^n (n + C_{n-1})$ for $n = 1$ and $C_0 = 0$. Once T_1 is chosen, we choose x_1 such that $\|x_1\| = 4^{-1}$ and $\|T_1 x_1\| \geq \frac{2}{3} \|T_1\| \|x_1\|$. We next choose T_2 and then x_2 and so on. Now go ahead and complete the proof as in a text book.

151. The following version is more often used.

Corollary 15. *Let X, Y be as above and let $T_n \in L(X,Y)$ be such that $\lim T_n x$ exists for all $x \in X$. If $Tx := \lim T_n x$, then $T \in L(X,Y)$.* □

152. Note that the last item does **not** say that $T_n \to T$ in $BL(X,Y)$. Consider $T_n \colon \ell^2 \to \mathbb{K}$ given by $T_n(z) := z_n$ where $z = (z_k)$. Then $\lim_n T_n z = 0$ so that $T = 0$. But, $\|T_n - T\| = \|T_n\| = 1$.

Another example: Let $T_n \colon \ell^2 \to \ell^2$ be defined by $(T_n z)(k) := z(k+n)$. Then

$$\lim_n (T_n z)(k) = \lim_n z(k+n) = 0.$$

Hence $T_n z \to 0$ so that the limit operator is $T = 0$. Clearly, $\|T_n z\|_2 \leq \|z\|_2$. But $T_n(e_k) = e_{k+n}$ so that $\|T_n\| = 1$.

153. Exercises.

(a) Let $a = (a_n)$ be a sequence such that $\sum_n a_n x_n$ converges for all $x \in \ell^1$. Show that $a \in \ell^\infty$.

(b) Let $a = (a_n)$ be a sequence such that $\sum_n a_n x_n$ converges for all $x \in \ell^2$. Show that $a \in \ell^2$.

(c) Formulate a general result which encompasses the last two exercises.

(d) Let $a = (a_n)$ be a sequence such that $\sum_n a_n x_n$ converges for all $x \in \mathbf{c_0}$. Show that $a \in \ell^1$.

(e) Let H_j, $j = 1, 2$ be Hilbert spaces. Let $A\colon H_1 \to H_2$ and $B\colon H_2 \to H_1$ be *any* maps such that $\langle Ax, y\rangle = \langle x, By\rangle$ for all $x \in H_1$ and $y \in H_2$. Show that A and B are linear and they are continuous.

A special case: If $T\colon H \to H$ is a self-adjoint map on a Hilbert space H (that is, $\langle Tx, y\rangle = \langle x, Ty\rangle$ for all $x \in H$), then T is a bounded linear operator on H.

(f) Show that a subset $A \subset X$ of a normed linear space X is bounded iff $f(A)$ is bounded for each $f \in X^*$.

154. An important and standard application of UBP is to establish the existence of $f \in C(\mathbb{T})$ whose Fourier series does not converge at 0 to $f(0)$.

f $f \in L^1[-\pi, \pi]$, let $\widehat{f}(k) = \frac{1}{2\pi}\int_{-\pi}^{\pi} f(t)e^{-ikt}dt$. Let $s_n(f, x) = \sum_{|k|\leq n} \widehat{f}(k)e^{ikx}$. Then one knows that

$$s_n(f, x) = \frac{1}{2\pi}\int_{-\pi}^{\pi} f(t)D_n(x-t)dt,$$

where

$$D_n(t) = \sum_{|k|\leq n} e^{ikt} = \frac{\sin(n+1/2)t}{\sin t/2}.$$

Consider the linear functionals $\Lambda_n : \mathcal{C}[-\pi, \pi] \to \mathbb{C}$ given by $\Lambda_n(f) = s_n(f, 0)$. By the next item 155, $\|\Lambda_n\| = \int_{-\pi}^{\pi} |D_n|$. Use the inequality $|\sin t| \leq |t|$ for all $t \in \mathbb{R}$ and the substitution $u = (n+1/2)t$ to show that $\|\Lambda_n\| = \int_{-\pi}^{\pi} |D_n| \to \infty$. Hence conclude the result.

155. Let X and g be as above. Define a linear functional $\Lambda(f) := \int_0^1 g(t)f(t)$ for $f \in X$. Show that $\|\Lambda\| = \int_0^1 |g(t)|\,dt =: \|g\|_1$. *Hint:* The idea to prove $\|g\|_1 \leq \|\Lambda\|$ is to approximate s$gn\,g$ by a sequence of continuous functions. Observe the following:

$$\begin{aligned}
\int |g| &= \int |g|\frac{1+n|g|}{1+n|g|} \\
&= \int \frac{|g|}{1+n|g|} + \int g\frac{n\overline{g}}{1+n|g|} \\
&\leq 1/n + \Lambda\left(g\frac{n|\overline{g}|}{1+n|g|}\right).
\end{aligned}$$

156. A last application of UBP is the following result on bilinear maps.

157. Let X and Y be normed linear spaces. Let $B\colon X \times Y \to \mathbb{K}$ be a linear map which is separately continuous, that is, for each $x_0 \in X$, the linear map $y \mapsto B(x_0, y)$ and for each $y_0 \in Y$, the linear map $x \mapsto B(x, y_0)$ are continuous on Y and X respectively. Then B is continuous on $X \times Y$.

Proof. For a fixed $x \in X$, let $T_x(y) := B(x,y)$. Since T_x is continuous on Y, it can be extended uniquely to its completion $\overline{Y}$. The bilinear map B can also be extended to $X \times \overline{Y}$ which will remain separately continuous. So, we assume without loss of generality, that Y is a Banach space.

For each $y \in Y$, there exists $C_y > 0$ such that

$$|B(x,y)| \leq C_y \|x\|, \quad \text{for all } x \in X.$$

We shift our focus now on T_x to conclude that this inequality means that the family of operators $\{T_x : x \in X, \|x\| = 1\}$ is pointwise bounded: $|T_x(y)| \leq C_y \|x\|$. By UBP, it follows that the family $\{T_x : x \in X, \|x\| = 1\}$ is bounded din the operator norm. That is, there exists $M > 0$ such that

$$\|T_x\| \leq M \quad \text{for all } x \in X \text{ with } \|x\| = 1.$$

Since $T_{\alpha x} = \alpha T_x$, it follows that $\|T_x\| \leq M \|x\|$ for all $x \in X$.

We now complete the proof as we would in the case of product of sequences (multiplication is bilinear!).

$$\begin{aligned} |B(x,y) - B(x_0,y_0)| &= |B(x, y - y_0) - B(x, y_0) - B(x_0, y_0)| \\ &\leq |B(x, y - y_0) - B(x - x_0, y_0)| \\ &= |T_x(y - y_0)| + |B(x - x_0, y_0)| \\ &\leq M \|x\| \|y - y_0\| + C_{y_0} \|x - x_0\|. \end{aligned}$$

Clearly, these estimates establish the continuity of B at (x_0, y_0). □

158. Let $f\colon (X,d) \to (Y,d)$ be a map. Then f is an open map iff for any ball $B(x,r)$, there exists $\varepsilon > 0$ such that $B(f(x),\varepsilon) \subset f(B(x,r))$.

159. Let $T\colon X \to Y$ be a linear map of normed linear spaces. Then T is open iff there exists $r > 0$ such that $rB_Y \subset TB_X$.

160.

Lemma 16. *Let X be a Banach space and C be a closed convex, symmetric subset such that $\cup_1^\infty nC = X$. Then C is a neighborhood of $0 \in X$.*

Proof. By Baire category, there exists an n so that nC and hence C has nonempty interior, say, $x + rB_X \subset C$. Since C is symmetric and convex,

$$\frac{1}{2}B_X = -\frac{1}{2}x + \frac{1}{2}(x + rB_X) \subset C.$$

□

161.

Lemma 17. *Let X and Y be Banach spaces. Let $T\colon X \to Y$ be a bounded linear map* **onto** *Y. Then TB is a neighbourhood of 0 in Y.*

Proof. Let $V := TB_X$. Then $\overline{V}$ has the properties listed in Lemma 16 and hence

$$\overline{V} \supset rB_Y \text{ for some } r > 0. \tag{20}$$

We now use completeness of X to show that $\overline{V} \subset 2V$. Fix any $y_0 \in \overline{V}$. Choose $x_1 \in B_X$ such that $\|y_0 - Tx_1\| \leq r/2$ so that $y_0 - Tx_1 \in \frac{1}{2}rB_Y$. Since

$$\frac{1}{2}rB_Y \subset \frac{1}{2}\overline{V} = \overline{T(\frac{1}{2}B_X)},$$

we can find $x_2 \in \frac{1}{2}B_X$ such that

$$y_0 - T(x_1 + x_2) \in \frac{1}{4}rB_Y.$$

Thus we have a sequence $x_n \in X$ such that $\|x_n\| \leq 2^{-n+1}$ and such that

$$\|y_0 - T(x_1 + \cdots + x_n)\| \leq 2^{-n}r.$$

Then $x := \sum_{n=1}^{\infty} x_n$ is such that $Tx = y_0$. That is,

$$\overline{V} \subset 2V. \tag{21}$$

From Eq. 20 and Eq. 21 it follows that $T(B_X) = V \supset \frac{1}{2}rB_Y$. □

162.

Theorem 18 (Open Mapping Theorem)**.** *Let X and Y be Banach spaces. Let $T\colon X \to Y$ be a bounded linear map* **onto** *Y. Then T is an open map.*

Proof. Follows from the last two lemmas. □

163.

Theorem 19 (Bounded Inverse Theorem)**.** *A one-to-one bounded linear map from a Banach space onto another is a topological isomorphism, that is, a linear isomorphism which is also a homeomorphism.* □

164. The concept of an isomorphism between two normed linear spaces X and Y is that there there exists a linear isomorphism : $X \to Y$ which is simultaneously a homeomorphism, that is, T is an 'isomorphism' between the vector spaces X and Y as well as an 'isomorphism' between the topological spaces X and Y. Such an isomorphism is called a *topological linear isomorphism.*

165. In view of the last item, what bounded inverse theorem says is that any bijective continuous linear operator from X to Y is a topological linear isomorphism.

In algebra, a bijective homomorphism f between two algebraic objects A and B of the same kind is automatically an isomorphism, as the inverse f^{-1} is a homomorphism of the required kind.

Outside algebra, this rarely happens. You have already come across two instances where this happens. (i) As a consequence of open mapping theorem in complex analysis, any one-one holomorphic function from an open subset $U \subset \mathbb{C}$ onto V is an 'holomorphic isomorphism' in the sense that f^{-1} is holomorphic on V. (ii) Any bijective continuous map from a compact space X onto a Hausdorff space Y is a homeomorphism.

166. The following is sometimes known as Two-Norms Theorem.

Let X be a vector space and $\| \|_1$ and $\| \|_2$ complete norms on X. Assume that there is an $M > 0$ such that $\|x\|_1 \leq M\|x\|_2$. Show that these two norms are equivalent.

167. Let X and Y be Banach spaces and $T \in BL(X)$. Assume that the range $T(X)$ is closed in Y. Show that there exists $C > 0$ such that for all $y \in T(X)$, there exists $x \in X$ such that (1) $Tx = y$ and (2) $\|x\| \leq C\|Tx\| = \|y\|$.

Sketch: We use open mapping theorem. Assume $y \neq 0$. Let $\delta > 0$ be such that $B_Y(0, \delta) \subset TB_X$. Let $z \in B(X)$ be such that $Tz = \frac{\delta}{2\|y\|}y$. Take $x = \frac{2\|y\|}{\delta}z$ and $C = 2\delta^{-1}$.

168. Let $T: X \to Y$ be a map between topological spaces. Recall the graph of T is defined to be $G(T) := \{(x, Tx) : x \in X\}$. We say that T is *closed* or T has *closed graph* if $G(T)$ is closed in $X \times Y$, where $X \times Y$ is given the product topology. In case X, Y are normed linear space, then $X \times Y$ is endowed with any of the equivalent product norms.

169.

Theorem 20 (Closed Graph Theorem)**.** *If X and Y are Banach spaces and $T: X \to Y$ is a linear map whose graph is closed in $X \times Y$, then T is bounded.*

Proof. Let π_i denote the projections of $X \times Y$ into the factors X and Y. Then $T = \pi_2 \circ \pi_1^{-1}$. Use the bounded inverse theorem. □

170. An immediate and useful corollary is the following.

Let X and Y be Banach spaces. A linear map $T: X \to Y$ is continuous iff it satisfies the condition: $x_n \to 0$ and $Tx_n \to y$ implies that $y = 0$.

171. The set of exercises in this Item is a list of some typical applications of OMT,BIT, Two-norms theorem and CGT.

(a) Use two-norms theorem to show that $(C([0, 1], \| \|_1)$ is not complete. *Hint.* If it is, then the identity map from $\| \|_\infty$ to $\| \|_1$ is a topological linear isomorphism. Hence we must be able find $C > 0$ such that $\|f\|_\infty \leq C\|f\|_1$. This is absurd. Think geometrically!

(b) Let $\| \|$ be any norm on $C[0, 1]$ with two properties: (i) It is complete and (ii) if $\|f_n - f\| \to 0$ then $f_n \to f$ pointwise. Show that $\| \|$ and $\| \|_\infty$ are equivalent. *Hint.* CGT

(c) Solve Item 153f assuming now X and Y are Banach. *Hint:* CGT.

(d) Solve Item 153e.

(e) X, Y, Z are Banach spaces. Assume that $A: X \to Y$ is a linear map and $B: Y \to Z$ is a bounded one-one linear operator. Show that A is bounded iff $B \circ A: X \to Z$ is bounded.

(f) Let A be a closed subspace of $\mathcal{C}[0, 1]$ and $g: [0, 1] \to \mathbb{R}$ be a function such that for all $f \in A$, $gf \in A$. Show that $M_g: A \to A$ given by $f \mapsto gf$ is a continuous linear map.

(g) Show that any separable Banach space is a quotient of ℓ^1. *Hint:* Choose a countable dense set $\{x_n\}$ in B_X. Define $T\colon \ell^1 \to X$ by setting $T(a_n) := \sum_n a_n x_n$. If $V := \ker T$, consider the canonical isomorphism $A\colon \ell^1/V \to X$. To show that A is onto, we show that T is onto. If $x \in B_X$, choose x_{n_1} such that $\|x - x_{n_1}\| < 1/2$. Since $\|2(x - x_{n_1})\| < 1$, we can find an n_2 etc. By induction, we have $\left\| x - \sum_{r=1}^k 2^{-r+1} x_{n_r} \right\| < 2^{-k}$. Think of a suitable preimage of x in ℓ^1.

(h) Let $A \in BL(X,Z)$ and $B \in BL(Y,Z)$. Assume that X, Y, Z are Banach spaces. Further assume that for any $x \in X$, there exists a unique $y \in Y$ such that $Ax = By$. Define $Tx = y$. Show that $T \in BL(X,Y)$.

(i) Let $(X, \mathcal{B}, \mu)$ be a σ-finite measure space. Assume that $\varphi\colon X \to \mathbb{K}$ be measurable. Let $1 \le p \le \infty$. Let M_φ denote the multiplication operator $f \mapsto \varphi f$. Assume that M_φ maps L^p to itself. Show (i) that M_φ is a continuous linear operator and (ii) that $\varphi \in L^\infty$. *Hints:* (i) Recall an important fact about the convergence of (f_n) in L^p for $1 \le p < \infty$. Use Item 170. (ii) Go through Item 112f.

(j) Let $T\colon L^2[0,1] \to L^2[0,1]$ be a continuous linear operator such that T maps $C[0,1]$ to itself. Show that $T\colon (C[0,1], \|\ \|_\infty) \to (C[0,1], \|\ \|_\infty)$ is continuous.

172. **Topological Complements.** This item deals with a standard application of closed theorem.

(a) Let Y be a closed subspace of a normed linear space X. We say that Y has a *topological complement* in X if there exists a closed subspace Z such that $Y \cap Z = (0)$ and $X = Y + Z$. In such a case, we also say that Y is a direct summand of X

(b) Y has a topological complement in X iff there exists a closed subspace Z such that any $x \in X$ can be written uniquely as $x = y + z$ with $y \in Y$ and $z \in Z$.

(c) Show that any closed vector subspace V of a Hilbert space H has a topological complement. *Hint:* Orthogonal complements exist.

(d) Topological complements need not be unique. (Two dimensional spaces will give examples!)

(e) If Y is a finite dimensional subspace of a normed linear space X, there is a closed subspace Z such that $Y \cap Z = (0)$ and $X = Y + Z$. That is, Y is a direct summand. *Hint:* If $\{y_i : 1 \le i \le n\}$ is a basis of Y, start with linear functionals f_i on Y with the property $f_i(y_j) = \delta_{ij}$. (This is a popular exercise all over the globe!)

(f) Let X be a Banach space and Y a closed subspace. Let Z be any topological complement of Y in X. Define $P\colon X \to Y$ by setting $Px = y$ where $x = y + z$ as above. Show that $P^2 = P$ and P is continuous. P is called the *projection* of X onto Y (with respect to the decomposition $X := Y \oplus Z$). *Hint:* To prove continuity of P, use Item 170.

(g) Y has topological complement in X iff there exists $P \in BL(X,X)$ such that $P^2 = P$ and $Y = \{x \in X : Px = x\}$.

(h) Let X be a Banach space and Y, Z closed linear subspaces of X such that $Y \cap Z = (0)$. Then $Y + Z$ is closed iff there is a $C > 0$ such that $\|y\| \le C\,\|y + z\|$ for $y \in Y$ and $z \in Z$.

(i) Let X, Y be Banach spaces and $T\colon X \to Y$ be a linear map. If the null space $N(T)$ (kernel of T) is closed then T is continuous. *Hint:* Consider $S\colon X/N(T) \to Y$ given by $S(x + N(T)) = T(x)$.

(j) It can be shown that $\mathbf{c_0}$ is not a direct summand of ℓ^∞. See, R. Whitley, Projecting m onto $\mathbf{c_0}$, Amer. Math. Monthly, **73**(1966), 285-286.

173. The next few items prove the very basic and useful results concerning compact operators on a Banach space. They are collectively known as Riesz-Schauder-Fredholm theory.

174. A linear operator $T \in L(X, Y)$ is *compact* if $\overline{T(B)}$ is compact where B is the unit ball in X. A compact linear map is necessarily continuous.

175. Let $T \in L(X, Y)$. Then T is compact iff the following holds: If (x_n) is a bounded sequence in X, then (Tx_n) has a convergent subsequence.

176. The identity map $I\colon X \to X$ is compact iff X is finite dimensional.

177. Any continuous linear map $T\colon X \to Y$ with $\dim(TX) < \infty$ is compact. Such operators are said to be of finite rank.

178. Let T denote the inclusion of $(C^1[0, 1], \| \ \|_{C^1})$ into $(C[0, 1], \| \ \|_\infty)$. Then T is compact.

179. Let $s \in \mathbb{R}$. Let

$$H_s := \{u\colon \mathbb{Z} \to \mathbb{C} : \sum_{n\in\mathbb{Z}} |u(n)|^2(1 + |n|^2)^s < \infty\}.$$

Show that H_s can be thought of as a Hilbert space with the norm

$$\|u\|_s := \left(\sum_n |u(n)|^2(1 + |n|^2)^s\right)^{1/2}.$$

The natural inclusion of $H_t \hookrightarrow H_s$ for $s < t$ is a compact linear operator. H_s is called the s-th Sobolev space. *Hint:* Observe that H_s is the L^2 space on $\mathbb{Z}$ w.r.t. the weighted measure $\mu(n) := (1 + |n|^2)^s$. To show the compactness of the inclusion, we show that the unit ball B_t of H_t is totally bounded in H_s. Let $u := \sum u_n e^{inx}$ be in the unit ball of H_t. For N to be chosen later,

$$\begin{aligned}
\sum_{|n|>N} |u_n|^2(1 + |n|^2)^s &\leq \sum_{|n|>N} |u_n|^2(1 + |n|^2)^t(1 + |n|^2)^{s-t} \\
&\leq \sum_{|n|>N} |u_n|^2(1 + |n|^2)^t(1 + |N|^2)^{s-t} \\
&< \varepsilon,
\end{aligned}$$

if N is chosen sufficiently large.

The first N terms lie in a finite dimensional space of v's such that if $v_n \neq 0$ then $|n| \leq N$. Since any finite dimensional normed linear space is locally compact, we can find an ε-net for such a set. Thus the full unit ball in H_t is totally bounded in H_s.

180. Let $X = C([a,b])$ and $K \in C([a,b] \times [a,b])$. We define

$$T_K f(x) := \int_a^b K(x,y) f(y) dy.$$

Show that T_K is a compact linear operator with $\|T\| \le (b-a)\sup_{(x,y)} |K(x,y)|$. T_K is called *Fredholm integral operator* with kernel K. *Hint:* Recall Arzela-Ascoli.

181. Let $\triangle$ be the triangle $a \le x \le b$, $a \le y \le x$ in $\mathbb{R}^2$. Let $K \in C(\triangle)$. Show that the map $f \mapsto Tf$ where $Tf(x) = \int_0^x K(x,y)f(y)dy$ is compact.

182. Let $A\colon X \to Y$ and $B\colon Y \to Z$ be compact linear operators between normed linear spaces. Then $B \circ A$ is compact, if one of A, B is compact.

Hence, the set $K(X)$ of compact linear operators on a normed linear space X is a is a two-sided ideal in $BL(X)$.

183. Assume that $T \in BL(X)$ is a topological linear isomorphism and compact. Then $\dim X < \infty$. In particular, if λI is compact, $\lambda \ne 0$, then $\dim X < \infty$.

184. Let $T \in BL(X)$ be compact. Let $P(X)$ be a polynomial such that $P(0) \ne 0$. Assume that $P(A)$ is compact. Show that $\dim(X) < \infty$.

185. Let X be Banach. Let $T_n \in BL(X)$ be compact and assume that $T_n \to T$ in $BL(X)$. Then T is compact. *Hint:* Enough to show that $T(B_X)$ is totally bounded. That is, you need to estimate $\|Tx\|$ for $x \in B_X[0.1]$. Observe that

$$\|Tx\| \le \|Tx - T_n x\| + \|T_n x\|.$$

Given $\varepsilon > 0$, choose N so that $\|T_N - T\| < \varepsilon/2$ and use the fact that $T_N(B_X)$ is totally bounded to find an $\frac{\varepsilon}{2}$-net. Verify that the $\frac{\varepsilon}{2}$-net for $T_N(B_X)$ is an ε-net for TB_X.

186. In view of the last item and Item 182, the set of compact operators in $BL(X)$ is a closed two-sided ideal.

187. Let H be a separable (infinite dimensional) Hilbert space with $\{e_n : n \in \mathbb{N}\}$ as an O.N.basis. Let (λ_n) be a bounded sequence in $\mathbb{K}$. Define a diagonal operator T on H such that $Te_n = \lambda_n e_n$. Then T is continuous and $\|T\| = \sup\{|\lambda_n|\}$. Show that T is compact iff $\lim \lambda_n = 0$. *Hint:* Observe that T_n defined by $T_n(e_k) = \lambda_k e_k$ for $1 \le k \le n$ and $T_n e_k = 0$ for $k > n$ is of finite rank and that $\|T - T_n\| = \sup\{|\lambda_k| : k > n\}$. If T is compact and $\lim \lambda_n \ne 0$ then there exists $\varepsilon > 0$ such that for an infinite subset $A \subset \mathbb{N}$, we have $|\lambda_k| > \varepsilon$ for $k \in \mathbb{N}$. Show that $\|Te_j - Te_k\|^2 > 2\varepsilon^2$ for distinct $j, k \in A$.

188. Let us recall a simple fact from linear algebra. Let V be a finite dimensional vector space over a field k. Let $T\colon V \to V$ be a linear map and $\lambda \in k$. Then the following dichotomy holds:
(i) $\ker(T - \lambda I)$ is nonzero, that is, λ is an eigenvalue of T.
(ii) $T - \lambda I$ is a bijection.
(Dichotomy: exactly one and only one will be true. Recall trichotomy in $\mathbb{R}$.)

For a compact operator T on a Banach space X almost the same dichotomy is true, provided that we assume $\lambda \ne 0$. Of course, the second conclusion is $(T - \lambda I)^{-1}$ is a topological linear isomorphism. Why is the condition $\lambda \neq 0$ needed? If the dichotomy holds for $\lambda = 0$ also, then T^{-1} will be a continuous linear operator and hence $I = T \circ T^{-1}$ will be compact. Therefore X must be finite dimensional.

189. This items culminates in the most basic result called Fredholm alternative/dichotomy.

Lemma 21. *Let X be a Banach space. Assume that $T \in BL(X)$ is compact and that $\lambda \in \mathbb{C}^*$ is not an eigenvalue of T. Then there exists $C > 0$ such that for all $x \in X$ we have*

$$\|(T - \lambda I)x\| \geq C\,\|x\| \text{ for } x \in X. \tag{22}$$

Proof. If no such C exists, then there exists a sequence (x_n) of unit vectors such that $Tx_n - \lambda x_n \to 0$. Since T is compact there exists a subsequence of (Tx_n) which is convergent. After re-indexing, let us assume that $Tx_n \to y$. Let $y_n := Tx_n$ so that $y_n \to y$. Observe that

$$(T - \lambda I)Tx_n = T(T - \lambda I)x_n \to 0.$$

That is, $(T - \lambda I)y_n \to 0$. Since $y_n \to y$, it follows that $Ty_n - \lambda y_n \to 0$ or $Ty - \lambda y = 0$. Hence λ has an eigenvector, provided y is nonzero. Can y be zero? If it is, the facts that $Tx_n \to 0$ and that $Tx_n - \lambda x_n \to 0$ imply $\lambda x_n \to 0$. That is, $|\lambda| = |\lambda|\,\|x_n\| \to 0$. This contradicts our assumption that $\lambda \neq 0$. ☐

Corollary 22. *Keep the notation of the last lemma. Then the range $R(T - \lambda I)$ is closed.*

Proof. Let $(T - \lambda I)x_n \to y$. From (22), it follows that (x_n) is Cauchy. Since X is complete, (x_n) converges to some x. Clearly, $(T - \lambda I)x = y$. ☐

190. An analogue of the lemma in Item 189.

Lemma 23. *Let X, Y be Banach spaces. Let $T, A \in BL(X, Y)$. Assume that A is compact and that there exists $C > 0$ such that*

$$\|x\| \leq C\,\|TX\| + \|Ax\|, \quad x \in X.$$

Then the range $T(X)$ is closed in Y.

Proof. The intuitive content of the lemma is that if T is 'invertible' modulo compact operators, then its range is closed.

Let $Z := \ker T$. Let (x_n) be such that $Tx_n \to y$. We need to show the existence of x such that $Tx = y$.

Case 1: Assume that $(d(x_n, Z)) = (\|x_n + Z\|)$ is bounded. In this case we may choose $x_n' \in x_n + Z$ such that $T(x_n') = T(x_n)$ and (x_n') is bounded. So, without loss of generality, assume that (x_n) is bounded. Again, using the compactness of A, we assume without loss of generality that (Ax_n) is convergent. The inequality of the lemma implies

$$\|x_n - x_m\| \leq C\,\|Tx_n - T_m\| + \|Ax_n - Ax_m\| \to 0.$$

Since X is complete, $x_n \to x$. Clearly, $Tx = y$.

Case 2: Assume that $(d(x_n, Z)) = (\| x_n + Z \|)$ is not bounded. We show that this cannot arise. Assume, if possible, that $d(x_n, Z) \to \infty$. Assume that $d(x_n, Z) \geq 2$ for all n. We select $v_n \in x_n + Z$ so that

$$d(x_n, Z) \leq d(v_n, Z) \leq 1 + d(x_n, Z).$$

Let $u_n := v_n / \| v_n \|$. We claim that $d(u_n, Z) \geq 1/3$. For,

$$\| u_n + Z \| \equiv d(u_n, Z) = \frac{1}{\| v_n \|} \| v_n + Z \| = \frac{1}{\| x_n \|} \| v_n + Z \| .$$

Hence the last term on the RHS is

$$\frac{d(x_n, Z)}{1 + d(x_n, Z)} = 1 - \frac{1}{1 + d(x_n, Z)} \geq 1/3.$$

Using compactness of A, by passing to a subsequence, we assume that (Au_n) converges. Note that

$$T(u_n) = \frac{1}{\| v_n \|} T(v_n) = \frac{1}{\| v_n \|} T(x_n) \to 0,$$

as $Tx_n \to y$. It follows that (u_n) is Cauchy:

$$\| u_n - u_m \| \leq \| Tu_n - Tu_m \| + \| Au_n - Au_m \| \to 0.$$

Hence $u_n \mathbf{t} u$ with $\| u \| = 1$. Since $d(u_n, Z) \geq 1/3$ it follows that $d(u, Z) \geq 1/3$. But, $Tu = \lim Tu_n = 0$ so that $u \in Z$, a contradiction! □

Theorem 24 (Fredholm Alternative)**.** *Let X be a Banach space and $T \in BL(X)$ compact. Let $\lambda \in \mathbb{C}^*$. Then exactly one of the following is true.*
(i) *λ is an eigenvalue of T so that* $\ker(T - \lambda I)$ *is nonzero.*
(ii) *$T - \lambda I$ is a topological isomorphism, that is,* $(T - \lambda I)^{-1} \in BL(X)$.

Proof. Assume that λ is not an eigenvalue of T. In view of BIT, it suffices to show that $(T - \lambda I)$ is bijective. It is one-one thanks to (22). So, we need only establish that $(T - \lambda I)$ is onto. Assume that this does not happen.

Let $X_1 := (T - \lambda I)X$. Observe that (22) shows that $(T - \lambda I)$ is one-one. Since by the last corollary, X_1 is closed and hence complete. Thus, $T - \lambda I$ is a continuous linear bijection of the Banach space X onto the Banach space X_1. Hence by BIT, $T - \lambda I$ is a topological linear isomorphism of X onto X_1. Let $X_{n+1} := (T - \lambda I)X_n$, $n \geq 1$. Then by the argument above, we see that (i) each X_{n+1} is a proper closed linear subspace of X_n and that $T - \lambda I$ is a topological linear isomorphism of X_n onto its proper closed subspace X_{n+1}. Hence by Riesz lemma, there exist a unit vector $x_n \in X_n$ such that $d(x_n, X_{n+1}) \equiv \inf\{\| x_n + y \| : y \in X_{n+1}\} \geq 1/2$.

We now claim that the sequence (Tx_n) cannot a convergent subsequence. For, since $(T - \lambda I)x_n \in X_{m+1}$ and $(T - \lambda I)x_m \in X_{m+1}$, we obtain $Tx_n - \lambda x_n - (Tx_m - \lambda x_m) \in X_{m+1}$ so that

$$Tx_n - Tx_m \in -\lambda x_m + (\lambda x_n + X_{m+1}) = -\lambda x_m + X_{m+1}.$$

Since $d(x_m, X_{m+1}) = \| x_m + X_{m+1} \| \geq 1/2$, it follows that $\| -\lambda x_m + X_{m+1} \| \geq \frac{|\lambda|}{2}$. Therefore, $\| Tx_n - Tx_m \| \geq \frac{|\lambda|}{2}$. The claim is proved. This contradicts out assumption that T is compact. □

191. Let X be a Banach space over $\mathbb{C}$. Let $T \in BL(X)$. We let

$$\text{Spec}\,(T) := \{\lambda \in \mathbb{C} : (T - \lambda I)^{-1} \notin BL(X)\}$$

Spec (T) is called the *spectrum* of T. Let us look at some typical cases when $\lambda \in \mathbb{C}$ lies in Spec (T).

(a) If $\dim(X) < \infty$, then $\lambda \in \text{Spec}\,(T)$ iff λ is an eigenvalue of T.

(b) If T is a compact operator on an *infinite dimensional* normed linear space X, then $0 \in \text{Spec}\,(T)$. For, otherwise, $T^{-1} \in BL(X)$. Hence $I = T \circ T^{-1}$ is a compact operator. We conclude that $\dim X < \infty$.

(c) In item 121, we solved an integral equation. The integral operator T is compact and what we achieved there shows that $\text{Spec}\,(T) = \{0\}$. (See Item 194.) Contrast this with the case of an operator on a finite dimensional space *over* $\mathbb{C}$. Its spectrum contains exactly as many elements (counting with multiplicity) as the dimension of the space.

192. We now prove the spectral theorem for a compact operator on a Banach space. We need a definition.

Theorem 25 (Spectral Theorem for Compact Operators)**.** *Let X be an infinite dimensional Banach space. Let T be a nonzero compact operator on X. Then the following hold:*
(i) $0 \in \text{Spec}\,(T)$.
(ii) *The spectrum* Spec (T) *is countable.*
(iii) *If we enumerate* Spec (T) *as a sequence* (λ_n)*, then* $\lim \lambda_n = 0$.
(iv) *If* Spec (T) *is infinite, then* 0 *is the only cluster point of* Spec (T).
(v) *For each nonzero* $\lambda \in \text{Spec}\,(T)$*,* $\dim \ker(T - \lambda I) < \infty$.

Proof. (i) is already observed. If we prove that for any $\varepsilon > 0$, the set $A_\varepsilon := \{\lambda \in \text{Spec}\,(T) : |\lambda| \geq \varepsilon\}$ is finite, then (ii),(iii) and (iv) follow. (Why does (ii) follow? Look at $A_{1/n}$, then Spec (T) is 'essentially' their union. One of them has to be uncountable.)

Suppose there exists $\varepsilon > 0$ such that A_ε is infinite. Then we can find a sequence (λ_n) in A_ε such that λ_n's are distinct. Since T is compact each λ_n is an eigenvalue. Hence there exist unit vectors x_n such that $Tx_n = \lambda_n x_n$. The set $\{x_n : n \in \mathbb{N}\}$ is linearly independent. (Can you recall the proof?) Our idea now is to mimic the last part of the proof of Fredohlm alternative. If we let X_n to be linear span of $\{x_k : 1 \leq k \leq n\}$, then X_n is a closed (why?) proper subspace of X_{n+1} for each $n \in \mathbb{N}$.

By Riesz lemma, we can find a sequence (u_n) of unit vectors such that $d(u_n X_{n-1}) \geq 1/2$ for $n \geq 2$. To exploit compactness of T, we need to find a bounded sequence (v_n) such that (Tv_n) has no convergent subsequence. As we want to use $d(u_n, X_{n-1}) \geq 1/2$, we start with $v_n := \lambda_n^{-1} u_n$. We would like to show that $\|Tv_n - Tv_m\|$ is bounded below by some positive constant. If we show that $Tv_n - Tv_m = u_n + z$ where $z \in X_{n-1}$, it will follow that $\|Tv_n - Tv_m\| \geq 1/2$. A little calculation shows that

$$z = Tv_m + u_n - \frac{1}{\lambda_n} Tu_n = Tv_m + \frac{1}{\lambda_n}\left(\lambda_n I - T\right) u_n.$$

The stuff inside the brackets is a linear combination of $\{x_1, \ldots, x_{n-1}\}$. Hence $z \in X_{n-1}$. Consequently, $\|Tv_n - Tv_m\| = \|u_n - z\| \geq d(u_n, X_{n-1}) \geq 1/2$. It follows that (Tv_n) cannot have a convergent subsequence, which contradictions the fact that T is compact.

(v). Note that by Fredohlm alternative, any nonzero $\lambda \in \operatorname{Spec}(T)$ is an eigenvalue. Let $X_\lambda := \ker(T - \lambda I)$ be th eigenspace of T corresponding to the eigenvalue λ. Then T maps X_λ to itself and its restriction to X_λ is λI. Since the restriction of any compact operator T to any vector subspace is compact, it follows that λI is compact on X_λ and hence $\dim X_\lambda < \infty$. $\square$

193. We look at an integral operator and determine its spectrum. Consider

$$T\colon (C([0,1]), \|\ \|_\infty) \to (C([0,1]), \|\ \|_\infty), \quad \text{where } Tf(x) := \int_0^{1-x} f(t)\, dt.$$

We claim that $\operatorname{Spec}(T) = \{0\} \cup \{\lambda : \frac{1}{\lambda} = \frac{\pi}{2} + 2k\pi, k \in \mathbb{Z}\}$. Since we know that T is compact, we need only find the nonzero eigenvalues of T, that is we need to solve the integral equation

$$\int_0^{1-x} f(t)\, dt = \lambda f(x), \lambda \neq 0.$$

The basic idea is to show that a solution of the integral equation is a solution of an ODE which can be easily determined.

We now work towards this end. First, some observations. If f is a solution of the integral equation above, then f is C^1 and $f(1) = 0$. Differentiating the integral equation, we obtain $\lambda f'(x) = -f(1-x)$ so that we conclude that f is C^2 and $f'(0) = 0$. Also, $\lambda f'(1) = -f(0)$. Once more differentiating, we obtain $g''(x) = -\frac{1}{\lambda^2} g(x)$. The solutions of this ODE with the 'initial condition' $g'(0) = 0$ are $A\cos(x/\lambda)$. In order that they satisfy the further conditions $f(1) = 0$ and $f'(1) = -f(0)$, λ must satisfy $\frac{1}{\lambda} = \frac{\pi}{2} + 2k\pi$.

194. Let K be a continuous real valued function on $[a,b] \times [a,b]$. Let T be the Volterra integral operator defined by

$$Tf(x) := \int_a^x K(x,y) f(y)\, dy.$$

Note the upper limit in the integral. Let $M > 0$ be such that $|K(x,y)| \leq M$. We show that the kernel K_n of T^n is a Volterra kernel (i.e. $K(x,y) = 0$ for $x < y$) and satisfies the inequality

$$|K_n(x,y)| \leq \frac{M^n (x-y)^{n-1}}{(n-1)!}, \qquad x \geq y$$

Assume that these statements are true for K_n. Then

$$K_{n+1}(x,y) = \int K(x,z) K_n(z,y)\, dz.$$

The integral vanishes at z unless $z < x$ and $z > y$ and hence K_n is a Volterra kernel.

We also have

$$\begin{aligned}
|K_{n+1}(x,y)| &\leq \int_y^x |K(x,z)K_n(z,y)|\,dz \\
&\leq \frac{M^{n+1}}{(n-1)!}\int_y^x (z-y)^{n-1}\,dz \\
&= \frac{M^{n+1}}{n!}(x-y)^n.
\end{aligned}$$

It follows that $\|T^n\| \leq \frac{M^n(b-a)^n}{(n-1)!}$. Also the estimate for T^n shows that the series $-\sum_{n=0}^{\infty}\lambda^{-(n+1)}T^n$ converges (for $\lambda \neq 0$) to $(T-\lambda I)^{-1}$. Hence $\operatorname{Spec} T = \{0\}$.

195. We look at a third integral operator and determine its spectrum. Let $K(s,t) = \min\{s,t\}$ for $(s,t) \in [0,1]\times[0,1]$. Let T be the corresponding kernel operator on $(C[0,1], \|\ \|_\infty)$. Note that

$$g(x) := Tf(x) = \int_0^x yf(y)\,dy + x\int_x^1 f(y)\,dy.$$

As in Item 193, we conclude that g is C^2 and we have

$$g(0) = 0 = g'(1) \quad \text{and} \quad g'' = -f$$

Note also that,

$$\int_0^1 Tf(x)f(x)\,dx = \int_0^1 g(x)(-g''(x))\,dx = -\left[g(x)g'(x)\right]_0^1 + \int_0^1 (g'(x))^2\,dx.$$

Using the conditions $g(0) = 0 = g'(1)$, we see that $\langle Tf, f\rangle_{L^2} \geq 0$. Hence if λ is an eigenvalue of T, then $\lambda \geq 0$.

The solutions of $f'' = -\lambda f$ with $f(0) = 0 = f'(1)$ are $C\sin(\frac{x}{\sqrt{\lambda_n}})$ where $\lambda_n^{-1} = (\frac{\pi}{2}+n\pi)^2$. Check the calculations

Hence the $\operatorname{Spec}(T) = \{0\}\cup\{(\frac{\pi}{2}+n\pi)^{-2} : n \in \mathbb{Z}\}$.

As is customary in books on Functional Analysis, we shall assume that our Hilbert spaces are over $\mathbb{C}$.

196. Let H be a Hilbert space over $\mathbb{C}$. Let $T \in BL(H)$, $\lambda \in \mathbb{C}$ is called an *eigen value* of T if

$$H_\lambda := \{x \in H \mid Tx = \lambda x\}$$

is non-zero. If λ is an eigenvalue of T, then H_λ is called the eigenspace of T corresponding to the eigenvalue λ.

Till Item ????, the following notation holds: If $T \in BL(H)$ is a self-adjoint compact operator, then $\Lambda(T)$ denotes the set of eigenvalues of T.

197. The eigenvalues of a self-adjoint operator $T \in BL(H)$ are real.

198. Let $T \in BL(H)$ be self-adjoint and λ, μ are eigen alues of T. Assume $\lambda \neq \mu$. Then $H(\lambda) \perp H(\mu)$.

199. The crucial point in the spectral theorem for self-adjoint operators on $\mathbb{C}^n$ is the existence of an eigenvalue. It is usually proved by an appeal to the fundamental theorem of algebra. We give a more analytic proof which generalizes to compact self-adjoint operators on any Hilbert space.

Proposition 26. *Let H be finite dimensional and $T\colon H \to H$ be and self-adjoint. Then one of the numbers $\pm \|T\|$ is an eigenvalue of T.*

Proof. There exists an $x \in H$ with $\|x\| = 1$ and $\|T\| = \|Tx\| = \lambda$, thanks to the compactness of the unit sphere of H. We have

$$\begin{aligned} \|T\|^2 = \lambda^2 = \langle Tx, Tx\rangle &= \left\langle T^2x, x\right\rangle \\ &\leq \left\|T^2\right\| \quad \text{by Cauchy-Schwarz Inequality} \\ &\leq \|T\|^2 \leq \|T\|^2 . \end{aligned}$$

Thus equality holds everywhere. By the equality part of the Cauchy-Schwarz inequality, we see that T^2x must be a scalar multiple of x, say, $T^2x = \alpha x$. Since $\lambda^2 = \left\langle T^2x, x\right\rangle = \alpha$, it follows that

$$(T + \lambda I)(T - \lambda I)x = 0.$$

The result follows. □

200. We now prove the following.

Proposition 27. *Let $T\colon H \to H$ be compact and self-adjoint. Then one of the numbers $\pm \|T\|$ is an eigen value.*

Proof. The idea is to adapt the proof for the dimensional case.

Choose $x_n \in H$ such that $\|x_n\| = 1$ and $\|Tx_n\| \to \|T\|$. We have

$$\begin{aligned} \left\langle T^2x_n - \|Tx_n\|^2 x_n, T^2x_n - \|Tx_n\|^2 x_n\right\rangle &= \left\|T^2x_n\right\|^2 - 2\|Tx_n\|^4 + \|Tx_n\|^4 \\ &\leq \|T\|^2\|Tx_n\|^2 - \|Tx_n\|^4 \to 0, \end{aligned}$$

as $n \to \infty$. Thus, $T^2x_n - \|Tx_n\|^2 x_n \to 0$ and hence

$$T^2x_n - \|T\|^2 x_n \to 0.$$

Since T^2 is compact, (T^2x_n) has a convergent subsequence. For simplicity sake, let us assume that (T^2x_n) converges to y. It follows that $x_n \to x$ for some x. We deduce that $T^2x - \|T\|^2 x = 0$. The proof is completed as earlier. □

201. Show that $H = \oplus_{\lambda\in\Lambda(T)} H(\lambda)$, where

$$\bigoplus_{\lambda\in\Lambda(T)} H(\lambda) := \left\{ x \in H \;\middle|\; x = \sum_{\lambda} x_\lambda, x_\lambda \in H(\lambda) \text{ with } \sum \|x_\lambda\|^2 < \infty \right\}$$

is the closure of the span of $H(\lambda)$'s. *Hint:* If $H \neq \oplus H(\lambda)$, then restrict T to $(\oplus H_\lambda)^\perp$ and apply Prop. 27.

202. The results of Items ??? lead to a proof of the **Spectral Theorem** for compact self-adjoint operators:

Theorem 28 (Spectral Theorem for Compact Self-Adjoint Operators)**.** *Let H be a non-zero Hilbert space and T a compact self-adjoint operator on H. Let $\Lambda(T)$ denote the set of eigen values of T. Then*
(i) One of the numbers $\pm \|T\|$ is an eigen value of T.
(ii) $\Lambda(T)$ has no accumulation points except possibly $\lambda = 0$.
(iii) If $H(\lambda) := \{x \in H \mid Tx = \lambda x\}$, then $\dim H(\lambda)$ is finite for $\lambda \neq 0$.
(iv) $H = \oplus_{\lambda \in \Lambda(T)} H(\lambda)$, the Hilbert space direct sum of eigen subspaces of T. That is, the closed linear span of the eigen subspaces of T is H. In particular, there exists a complete ON basis of H consisting of eigenvectors of T.
(v) If $\Lambda(T)$ is infinite, then 0 is a limit point of $\Lambda(T)$. □

Proof. To prove (v), observe that $\Lambda(T)$ is a subset of $\mathbb{R}$ contained in the interval $[-\|T\|, \|T\|]$. For, if $|\lambda| > \|T\|$, then $(T - \lambda I)^{-1}$ exists and hence λ cannot be an eigenvalue. Now the result follows form Bolzano-Weierstrass and (iv) of the theorem in Item 192. □

203. Let $T\colon \ell^2 \to \ell^2$ be defined by $A((x_k)) = \left(\frac{x_k}{k}\right)$. Show that T is a compact self-adjoint operator. Find the spectral decomposition of T, i.e., find the eigenvalues of this operator, corresponding eigenspaces and the projections of ℓ^2 onto their spaces and write $A = \sum_{\lambda_j \in \Lambda(T)} \lambda_j P_j$.

204. **Adjoint of an operator.** Let X and Y be normed linear spaces. Let $T \in BL(X, Y)$. We define the *adjoint* $T^*\colon Y^* \to X^*$ by the following equation:

$$T^*(g)(x) := g(TX).$$

We write the above equation by employing a prevalent practice in algebra:

$$(T^*(g), x) = (g, T(x)).$$

It is easy to show that T^* is a bounded linear operator.

205. Let X, Y and Y be normed linear spaces. The following are true:
(a) If $T \in BL(X, Y)$, then $T^* \in BL(Y^*, X^*)$. We also have $\|T\| = \|T^*\|$.
(b) $(\alpha A + \beta B)^* = \alpha A^* + \beta B^*$ for $A, B \in BL(X, Y)$ and $\alpha, \beta \in \mathbb{K}$.
(c) If $A \in BL(X, Y)$ and $B \in BL(Y, Z)$, then $(B \circ A)^* = A^* \circ B^*$.
(d) If $T \in BL(X, Y)$ is invertible (in $BL(X, Y)$), then $T^*$is invertible in $BL(Y^*, X^*)$.

Proof. It is easily seen that $\|T^*\| \leq \|T\|$. To show the other way inequality, given $\varepsilon > 0$, let $x \in B_X$ be such that $\|Tx\| \geq \|T\| - \varepsilon$. Then choose $g \in B_{Y^*}$ such that $g(TX) = \|Tx\|$. We have

$$\|T^*\| \geq \|T^*g\| \geq \|T^*g(x)\| = |g(Tx)| = \|Tx\| \geq \|T\| - \varepsilon.$$

(a) follows from this. □

206. If T^* is bounded below, then $\operatorname{Im} T$ is dense in Y.

207. Let X be a Banach space and Y, a normed linear space. For $T \in BL(X,Y)$, the following are equivalent:
(a) T is invertible.
(b) T^* is invertible.
(c) $\operatorname{Im} T$ is dense in Y and T is bounded below.
(d) T and T^* are both bounded below.

208. For $T \in BL(X,Y)$, $\|T^*\| = \|T\|$. Observe that, for $x \in B_X$ and $g \in B^* := B_{Y^*}$,

$$|T^*g(x)| = |g(Tx)| \leq \|Tx\| \leq \|T\|.$$

Hence $\|T^*\| \leq \|T\|$. To see the reverse inequality,w e shall show that $\|T^*\| \geq \|T\| - \varepsilon$ for any $\varepsilon > 0$. Given $\varepsilon > 0$, since $\|T\| = \sup\{\|Tx\| : \|x\| = 1\}$, there exists $x \in B_X$ such that $\|Tx\| \geq \|T\|_\varepsilon$. By Hahn-Banach, there exists $g \in B^*$ such that $g(Tx) = \|Tx\|$. We observe for $g \in B^*$ and x as above that

$$\|T^*\| \geq \|T^*g\| \geq |T^*g(x)| = |g(Tx)| = \|Tx\| \geq \|T\| - \varepsilon.$$

209. We need the following lemma for the next result.

Lemma 29. *Let X and Y be Banach spaces, $T \in BL(X,Y)$. Then T is invertible iff (a)* $\operatorname{Im}(T)$ *is dense in X and (b) there exists $C > 0$ such that*

$$\|Tx\| \geq C\|x\|, \quad \textit{for all } x \in X.$$

Proof. Non-trivial part: T is one-one. For, $\|x\| \leq (1/c)\|Tx\|$. Hence $T\colon X \to \operatorname{Im}(T)$ is a bijection. The algebraic inverse $T^{-1}\colon \operatorname{Im}(T) \to X$ is continuous by the given estimate. Therefore it extends uniquely to a continuous linear map (again denoted by T^{-1}) from the closure of $\operatorname{Im}(T)$, that is from Y to X. Let $y \in Y$. Then there exists a sequence (x_n) such that $Tx_n \to y$. Since $\|x_n - x_m\| \leq C^{-1}\|Tx_n - Tx_m\|$, it follows that (x_n) is Cauchy. Since X is Banach, (x_n) converges to, say, $x \in X$. Clearly, $y = Tx$. Thus $\operatorname{Im}(T) = Y$. □

210. Let X be a Banach space, Y any normed linear space. Then $T \in BL(X,Y)$ is invertible iff T^* is invertible.

Proof. Assume that T^* is invertible. We claim that T is onto. If not, $\operatorname{Im}(T)$ is proper vector subspace of Y. Hence there exists a unit vector $g \in Y^*$ such that $g = 0$ on $\operatorname{Im}(T)$. Thus, $g(Tx) = 0$ for all $x \in X$. This is same as saying that $T^*g = 0$ or $T^*g \in \ker T^*$. But $\ker(T^*) = \{0\}$. This contradiction establishes that $\operatorname{Im}(T) = Y$. To prove the invertibility of T, in view of the last lemma, we need only the lower bound estimate.

Let $x \in X$. By Hahn-Banach, there exists $f \in X^*$ such that $\|f\| = 1$ and $f(x) = \|x\|$. We have

$$\begin{aligned}\|x\| = f(x) &= (T^{*-1}T^*f)(x)\\ &= (T^{*-1}f)(Tx)\\ &\leq \left\|T^{*-1}f\right\|\|Tx\|\\ &\leq \left\|T^{*-1}\right\|\|Tx\|.\end{aligned}$$

Hence $\operatorname{Im}(T)$ is onto and we have $\|Tx\| \geq C\|x\|$ where $C^{-1} = \left\|T^{*-1}\right\|$. By the last lemma T is invertible. □

211. We prove Schauder's theorem.

Theorem 30. *Let $T\colon X \to Y$ be a compact linear operator. Then $T^*\colon Y^* \to X^*$ is compact.*

Proof. Let B^* be the closed unit ball in Y^*. We need to show that the closure of T^*B^* in X^* is compact. Since X^* is complete, it suffices to show that T^*B^* is totally bounded. This means that we need to estimate $\|T^*g\|$ for $g \in B^*$, which in turn means, we need to estimate $\|T^*g(x)\| = \|g(Tx)\|$ for $x \in B$, the unit ball in X. This gives us a a hint how to exploit the compactness of T. Since T is compact, TB is totally bounded. Given $\varepsilon > 0$, there exist $x_i \in B$, $1 \leq i \leq n$, such that $\{Tx_i : 1 \leq i \leq m\}$ is an ε-net for TB. Thus we obtain

$$\text{For each } x \in B,\ \exists i\ (1 \leq i \leq m),\ \text{such that } \|Tx - Tx_i\| < \varepsilon. \tag{23}$$

The idea is to approximate $T^*g(x) = g(Tx)$ by $g(Tx_i)$. So we consider the mapping

$$\varphi\colon Y^* \to \mathbb{K}^m, \quad \text{defined by } \varphi(g) := (g(Tx_1), \ldots, g(Tx_m)).$$

Then φ is linear, continuous and of finite rank and hence compact. Therefore, $\varphi(B^*)$ is totally bounded subset of $\mathbb{K}^m$. Hence for the given ε, we obtain

$$\text{For each } g \in B^*,\ \exists i\ (1 \leq j \leq m),\ \text{such that } \|\varphi(g) - \varphi(g_j)\| < \varepsilon. \tag{24}$$

Since for $z = (z_1, \ldots, z_m) \in \mathbb{K}^m$, $|z_i| \leq \|z\|$, $(1 \leq i \leq n)$, (24) implies

$$\text{For each } g \in B^*,\ \exists j\ (1 \leq j \leq m),\ \text{such that } |g(Tx_i) - g_j(Tx_i)| < \varepsilon,\ 1 \leq i \leq n. \tag{25}$$

We are now ready to show that $\{T^*g_j : 1 \leq j \leq n\}$ is a 3ε-net for T^*B^*. Given $g \in B^*$, choose j so that (24) and hence (25) hold. Let $x \in B$. Choose i so that (23) holds. With j and i so fixed, we get

$$\begin{aligned}
|g(Tx) - g_j(Tx)| &\leq |g(Tx) - g(Tx_i)| + |g(Tx_i) - g_j(Tx_i)| + |g_j(Tx_i) - g(Tx_i)| \\
&\leq \|g\|\,\|Tx - Tx_i\| + \varepsilon + \|g_j\|\,\|Tx_i - Tx\| \\
&< 3\varepsilon.
\end{aligned}$$

This establishes that T^*B^* is totally bounded. □

212. Let H be a Hilbert space with a countable orthonormal basis. Let T be a compact linear operator on H. Then there exists a sequence (T_n) of finite rank operators such that $T_n \to T$ in $BL(H)$.

Proof. Assume that $\dim H = \infty$. Let $\{e_n : n \in \mathbb{N}\}$ be a countable O.N. basis of H. Let H_n be the linear span of $\{e_k : 1 \leq k \leq n\}$. Then H_n is closed (why?) and hence $H = H_n \oplus H_N^\perp$ is an orthogonal direct sum. Let P_b be the orthogonal projection of H onto H_n. Then P_n is of norm 1. Let $T_n := P_n \circ T$. Then T_n is of finite rank.

We claim that $T_n x \to Tx$ for each $x \in H$. For,

$$T_n x = P_n T x = P_n \left(\sum_{k=1}^{\infty} \langle Tx, e_k \rangle \, e_k \right) = \sum_{k=1}^{n} \langle Tx, e_k \rangle \, e_k.$$

Hence we obtain $\| Tx - T_n x \|^2 = \sum_{k=n+1}^{\infty} | \langle Tx, e_k \rangle |^2 \to 0$ as $n \to \infty$. (Note that so far, we have not used the compactness of T!)

We now use this along with the compactness of T to conclude that $\| T - T_n \| \to 0$. We need to estimate $\| Tx - T_n x \|$ for $\| x \| = 1$. Let B be the unit ball in H. Since T is compact, TB is totally bounded. Given $\varepsilon > 0$, there exists $x_1, \ldots, x_k$ such that

$$\text{for each } x \in B, \; \exists j \; (1 \le j \le n) \; \text{ such that } \; \| Tx - Tx_j \| < \varepsilon. \tag{26}$$

Since $T_n \to Tx$ for each $x \in H$, the same is true for x_j, $1 \le j \le k$ Since these are finitely many, there exists $N \in \mathbb{N}$ such that

$$\forall j \; (1 \le j \le n) \; \forall n \ge N, \text{ we have } \| T_n x_j - T x_j \| < \varepsilon. \tag{27}$$

We are now ready to estimate $\| T_n x - Tx \|$ for $x \in B$, using (26)–(27):

$$\begin{aligned} \| T_n x - Tx \| &\le \| T_n x - T_n x_j \| + \| T_n x_j - T x_j \| + \| T x_j - Tx \| \\ &= \| P_n (Tx - Tx_j) \| + \| T_n x_j - T x_j \| + \| T x_j - Tx \| \\ &\le \| Tx - Tx_j \| + \| T_n x_j - T x_j \| + \| T x_j - Tx \| \\ &< 3\varepsilon. \end{aligned}$$

□

213. In the next few items, we shall identify the duals of some classical sequence spaces. The basic idea is simple. How do we find the duals, say, of $\mathbb{C}^n$ or of a (separable) Hilbert space?

In the case of $\mathbb{C}^n$, let $\{e_k : 1 \le \le n\}$ be the standard basis. Then we write $z = \sum_k z_k e_k$. If $f \colon \mathbb{C}^n \to \mathbb{C}$ is a (necessarily continuous) linear functional, then $f(z) = \sum_k z_k f(e_k)$. Hence if we let $a_k := \overline{f(e_k)}$ and $a = (a_1, \ldots, a_n) \in \mathbb{C}^n$, then $f(z) = \langle z, a \rangle$, the standard Hermitian inner product on $\mathbb{C}^n$. Thus, we "identify" f with $a \in \mathbb{C}^n$. Thus we have conjugate linear isomorphism from the dual of $\mathbb{C}^n$ to $\mathbb{C}^n$ which also preserves the norm, that is, the operator norm $\| f \|$ is $\| a \|$. (Note that we have not made serious use of the (automatic) continuity of f.)

This argument can be extended to a (complex) Hilbert space H with an O.N. basis $\{e_n : n \in \mathbb{N}\}$. Given $x \in H$, we know $x = \sum_k \langle x, e_k \rangle \, e_k$, the sum being convergent in H and $\| x \|^2 = \sum_k | \langle x, e_k \rangle |^2$. If $a := (a_k) \in \ell^2$, then $f_a(x) := \sum_k a_k e_k$ defines a linear functional with $\| f_a \| = \| a \|_2$. If f is continuous linear functional on H, then we have $f(x) = \sum_k \langle x, e_k \rangle \, f(e_k)$. let $a_k := f(e_k)$ and consider $a := \sum_k \overline{a_k} e_k$. Is $a \in H$? The answer is yes, if we can show that $\sum_k |a_k|^2 < \infty$, that is to show that $\sup_N \sum_{k=1}^N |a_k|^2 < \infty$. The idea here is to estimate $\sum |a_k|^2$ in terms of $\| f \|^2$. Let $z = \sum_k z_k e_k$ be of unit norm. Consider $x := \sum_{k=1}^N z_k e_k$. Then

$$f(x) = \sum_{k=1}^{N} z_k a_k = f_b(z) \text{ where } b := (a_1, \ldots, a_N, 0, \ldots).$$

Hence $|f_b(z)| \le |f(x)| \le \| f \|$. Since this is true for all unit vectors $z \in H$, it follows that $\| f_b \| := \left(\sum_{k=1}^N |a_k|^2 \right)^{1/2} \le \| f \|$. Since N was arbitrary, we get the desired result.

214. We shall now show that the dual of ℓ^p is isometrically isomorphic to ℓ^q for $1 \le p < \infty$ where $(1/p) + (1/q) = 1$.

Fix $1 < p < \infty$ and q be its conjugate index. Let $a = (a_n) \in \ell^q$. Define $f_a(x) := \sum_n a_n x_n$ where $x \in \ell^p$. By Hölder's inequality, $|f_a(x)| \le \| a \|_q \| x \|_p$ and hence $\| f_a \| \le \| a \|_q$. We claim that $\| f_a \| = \| a \|_q$. The trick is to find an $x \in \ell^p$ such that $f_a(x) = \sum_n |a_n|^q$. This means that we are on the look out for an x such that $f_a(x) = \sum_n a_n x_n = \sum_n |a_n|^q$. An obvious choice is to let $x_n := \frac{|a_n|^q}{a_n}$ (if $a_n \ne 0$ etc.) That is $x_n = \overline{\operatorname{Sign} a_n} |a_n|^{q-1}$. (Here the sign of a complex number z is defined as $\operatorname{Sign} z = \frac{z}{|z|}$ if $z \ne 0$ and 0 otherwise.) Does $x = (x_n) \in \ell^p$? That is, do we have $\sum_n |x_n|^p < \infty$? Observe that $\sum_n |x_n|^p = \sum_n |a_n|^{p(q-1)} = \sum_n |a_n|^q < \infty$. In particular, $\| x \|_p = \| a \|_q^{q/p}$. Now for this x, we have

$$f_a(x) = \sum_n |a_n|^q = \| a \|_q^q = \| a \|_q^1 \| a \|_q^{q-1} = \| a \|_q \| x \|_p, \text{ as } q - 1 = q/p.$$

Hence $\| f_a \| = \| a \|_q$ as claimed.

Next, let $f \colon \ell^p \to \mathbb{C}$ be a continuous linear functional. Define $a_n := f(e_n)$. We claim that $(a_n) \in \ell^q$. If the result is true, then $\sum_n |a_n|^q = \| f \|^q$. Hence the idea here is to estimate $\sum_{k+1}^N |a_n|^q$ in terms of $\| f \|$. Let $x \in \ell^p$ be of norm 1. Fix $N \in \mathbb{N}$. Let $z := (x_1, \dots, x_N, 0, 0 \dots)$. Note that $\| z \| \le \| x \| = 1$. We have

$$f(z) = \sum_{k=1}^N a_k x_k = f_b(x) \text{ where } b := (a_1, \dots, a_N, 0, 0, \dots) \in \ell^q.$$

We therefore obtain

$$|f_b(x)| = |f(z)| \le \| f \| \| z \| \le \| f \|.$$

Since this true for all $x \in \ell^p$ of unit norm, we deduce that $\| f_b \| \le \| f \|$. But for the first part, it follows that the norm of the continuous linear functional f_b is $\| b \|_q$. Thus we get $\left(\sum_{k=1}^N |a_k|^q \right)^{1/q} \le \| f \|$. This being true for all N, we deduce that $a \in \ell^q$.

215. We shall end our study of dual spaces of some classical Banach spaces of functions with a summary.

(a) **Riesz Representation Theorem for $C[0,1]$.** Let $\Lambda \in (\mathcal{C}[0,1], \| \, \|_\infty) \to \mathbb{R}$ be a continuous linear functional. Then there exists a (not unique) function g, of bounded variation on $[0,1]$ such that

$$\Lambda(f) := \int_0^1 f(t) dg(t) \text{ and } \| \Lambda \| = V_a^b(g).$$

Hint: Extend Λ to $(B[0,1], \| \, \|_\infty)$. Define $g(t) = \Lambda(\chi_{[0,1]})$. To show that g is of bounded variation, notice that $|g(x_i) - g(x_{i-1})| = g(x_i) - g(x_{i-1})\varepsilon_i$, where $\varepsilon_i = \operatorname{sgn}\, [g(x_i) - g(x_{i-1})]$. To show that $\Lambda(f) = \int f dg(t)$ consider $s_n := \sum_{r=1}^n f(r/n)[\chi_{r/n} - \chi_{(r-1)/n}]$ which tends to f uniformly. Here $\chi_t := \chi_{[0,t]}$.

(b) Let p, q be as above. Any continuous linear function Λ on $L^p(X, m, \mu)$ of a measure space (X, m, μ) is given by $\Lambda(f) = \int fgd\mu$ for a unique $g \in L^q(X, m, \mu)$. This is just for record sake. See any standard book on measure theory for a proof. Special cases are the sequence spaces ℓ^p.

(c) Let X be a compact Hausdorff space. Let Λ be a continuous linear functional on the Banach space $(\mathcal{C}(X), \| \|_\infty)$. Then there exists a unique regular Borel measure μ on X such that $\Lambda(f) = \int_X f d\mu$. This is also just for reference and known as the Riesz representation theorem for $C(X)$. See W. Rudin, *Real and Complex Analysis*.

(d) The following table summarizes the duals of the classical spaces of functions. The second column exhibits the way a continuous linear function on the space of the first column arises, the third column the norm of such a continuous linear functional and the fourth identifies what the dual of the space of the first is isometric to.

Space	Representation	Norm	Dual
$\mathbf{c}$	$a \lim x_n + \sum a_n x_n$	$\lvert a\rvert + \sum \lvert a_n\rvert$	ℓ^1
$\mathbf{c_0}$	$\sum a_n x_n$	$\sum \lvert a_n\rvert$	ℓ^1
ℓ^p	$\sum a_n x_n$	$(\sum \lvert a_n\rvert^q)^{1/q}$	ℓ^q
$C[a,b]$	$\int_a^b f(x) dg(x)$	$\int_a^b \lvert g(x)\rvert \, dx$	$BV[0,1]$
H, Hilbert space	$\langle x, v\rangle$	$\lVert v \rVert$	$\overline{H}$
$L^p(X, \mathcal{B}, \mu)$	$\int_X f(x) g(x) d\mu$	$(\int_X \lvert g\rvert^q)^{1/q}$	$L^q(X, \mathcal{B}, \mu)$

216. Let $T \colon \mathbf{c_0} \to \mathbf{c_0}$ be defined by $Tx := y = (y_n)$, where $y_n = x_n/n$ for $x = (x_n) \in \mathbf{c_0}$. Show that the adjoint $T^* \colon \ell^1 \to \ell^1$ is given by $T^*(y) = (y_n/n)$.

217. Let $T \colon \ell^1 \to \ell^1$ is given by $T(x) = (0, x_1, x_2, \ldots)$. Show that the adjoint $T^* \colon \ell^\infty \to \ell^\infty$ is given by $T^*(y) = (y_2, y_3, \ldots)$.

218. Let H be a Hilbert space. Let $T \in BL(H)$. We define the Hilbert space adjoint $T^* \colon H \to H$ of T is defined by the equation $\langle Tx, y\rangle := \langle x, T^* y\rangle$.

What does this mean? For a fixed $y \in H$, the map $f_y : x \mapsto \langle Tx, y\rangle$ is linear and continuous: $|f_y(x)| \leq \|T\| \, \|y\| \, \|x\|$. Hence by the Riesz representation theorem, there exists a unique $z \in H$ such that $f_y(x) = \langle x, z\rangle$. We denote this z by $T * y$. We claim that the map $y \mapsto T^* y$ is linear and continuous.

$$\begin{aligned} \langle x, T^*(y_1 + y_2)\rangle := f_{y_1+y_2}(x) &= \langle Tx, y_1 + y_2\rangle \\ &= \langle Tx, y_1\rangle + \langle Tx, y_2\rangle \end{aligned}$$

$$\begin{aligned} |\langle x, T^* y\rangle| &= |\langle Tx, y\rangle| \\ &\leq \|T\| \, \|x\| \, \|y\| . \end{aligned}$$

real linear isomorphism of H^* with H but not complex linear. For, if $f \leftrightarrow y$, then $\lambda f \leftrightarrow \overline{gl}y$. Hence the dual H^* is *complex conjugate* linear isomorphism.

Thus, if we want to be faithful to the earlier definition of an adjoint, the adjoint should be a map from H^* to H^* and *not* from H to H. But in the case of a Hilbert space, it is customary and expedient to use the Hilbert space adjoint, as we get 'all' the information with this gadget.

219. Let H be a complex Hilbert space. Then the following are true:
 (a) For $T \in BL(H)$, we have $\|T\| = \|T^*\|$.
 (b) $(\alpha A + \beta B)^* = \overline{\alpha} A^* + \overline{\beta} B^*$ for $\alpha, \beta \in \mathbb{C}$ and $A, B \in BL(H)$.
 (c) $(AB)^* = B^* A^*$ for $A, B \in BL(H)$.
 (d) $T^{**} = T$ for $T \in BL(H)$.
 (e) $\|T\|^2 = \|TT^*\| = \|T^*T\| = \|T^*\|^2$ for $T \in BL(H)$.

To prove (e), observe the following:

$$\|T\|^2 = \sup_{\|x\|=1} \|Tx\|^2 = \sup_{\|x\|=1} \langle Tx, Tx\rangle = \sup_{\|x\|=1} \langle T^*Tx, x\rangle \leq \|T^*T\| \leq \|T\|\,\|T^*\|.$$

220. For $T \in BL(H)$, we have $\ker T = (\operatorname{Im} T^*)^{\perp}$ and $\ker T^* = (\operatorname{Im} T)^{\perp}$.

221. Let $T \in BL(H)$. We say that T is *Hermitian* (resp. *skew-Hermitian*) if $T^* = T$ (resp. $T^* = -T$). Hermitian operators are also known as self-adjoint operators. If the scalar field is $\mathbb{R}$, then they are called symmetric.

$T \in BL(H)$ is said to be a *positive* operator if (1) T is self adjoint and (2) $\langle Tx, x\rangle \geq 0$ for all $x \in H$.

$T \in BL(H)$ is said to be *normal* if $TT^* = T^*T$.

222. The following are some of the standard exercises concerning these concepts.

 (a) Let H be a **complex** Hilbert space and $T \in BL(H)$ is such that $\langle Tx, x\rangle = 0$ for all $x \in H$. Then $T = 0$. *Hint:* Use the polarization identity:

$$\begin{aligned} 4\langle Tx, y\rangle &= \langle T(x+y), x+y\rangle - \langle T(x-y), x-y\rangle \\ &\quad + i\langle T(x+iy), x+iy\rangle - i\langle T(x-iy), x-iy\rangle. \end{aligned}$$

 The result is false in the case of real Hilbert spaces. *Hint:* Rotation by $\pi/2$ in the real Hilbert space $\mathbb{R}^2$.

 (b) A bounded linear operator $A\colon H \to H$ is hermitian iff $\langle Ax, x\rangle$ is real for all $x \in H$. *Hint:* Let $B = A - A^*$. Then $\langle Bx, x\rangle = 0$ for all $x \in H$.
 The result is false in the case of a real hilbert space.

 (c) In view of Ex. 222b, in the case of a complex hilbert space $T \in BL(H)$ is positive iff $\langle Tx, x\rangle \geq 0$.

 (d) The hermitian operators have the following properties:
 (a) If S and T are hermitian operators and if $a \in \mathbb{R}$, then aT and $S + T$ are hermitian.
 (b) If T_n is hermitian and $\|T_n - T\| \to 0$, then T is hermitian.
 (c) If $T \in BL(H)$, then $T = A + iB$, where A and B are hermitian.
 (d) If S and T are hermitian, ST is hermitian iff $ST = TS$.
 (e) If $T \in BL(H)$, then TT^* and T^*T are hermitian.

(e) (a) Show that any orthogonal projection is positive.
(b) For $T \in BL(H)$, the operators T^*T and TT^* are positive.
(c) If T is Hermitian (resp. skew-Hermitian) what can you say about αT where $\alpha \in \mathbb{C}$. Hint: Think of special cases for α!

(f) Let $P \in BL(H)$ be Hermitian and a projection, i.e. $P^2 = P$. Then P is an orthogonal projection. *Hint:* P projects onto the closed subspace $\operatorname{Im} P$.

(g) Let $P \in BL(H)$ be a projection. Then the following are equivalent:
  (a) P is an orthogonal projection.
  (b) P is Hermitian.
  (c) P is normal.
  (d) $\langle Px, x\rangle = \|Px\|^2$ for all $x \in H$.

(h) Let $T \in BL(H)$. Then
  (a) T is normal iff $\|Tx\| = \|T^*x\|$ for all $x \in H$.
  (b) If T is normal then

$$\ker T = \ker T^* = (\operatorname{Im} T)^{\perp} = (\operatorname{Im} T^*)^{\perp}.$$

(i) Consider the shift operator $S\colon \ell^2 \to \ell^2$ defined by $S((x_n)) = (0, x_1, x_2, \ldots)$. Then its adjoint is given by $S^*((y_n)) = (y_2 y x_3, \ldots)$.

(j) Investigate the conditions on the kernel K so that the integral operator T_K on $L^2[0,1]$ with the kernel K will be (i) hermitian, (ii) positive.

(k) If $A \in BL(H)$ is positive, then so are A^n for all $n \in \mathbb{N}$.